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# Nonlinear dissipative wave equations with space-time dependent potentials

Maisa Khader
*University of Tennessee*

To the Graduate Council:

I am submitting herewith a dissertation written by Maisa Khader entitled "Nonlinear dissipative wave equations with space-time dependent potentials." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Grozdena Todorova, Major Professor

We have read this dissertation and recommend its acceptance:

Accepted for the Council:

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Vice Provost and Dean of the Graduate School

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# Nonlinear Dissipative Wave Equations with Space–Time Dependent Potentials

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Maisa Khader

May 2009

# Dedication

To the memory of my husband, Dr. Salah M. Khader. To my wonderful son Mustafa.

# Acknowledgments

All the praises and thanks be to Allah (God), the most Gracious, the most Merciful, who gave me the strength, the power, and the hope that life has to continue.

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# Abstract

We study the long time behavior of solutions of the wave equations with absorption abs $(u(t,x))^{p-1}u(t,x)$ and variable damping $a(t,x)u_t(t,x)$, where $p$ belongs to $(1, n+2/n-2)$ and $a(t,x) \sim a_0(1+abs(x))^{-alpha}(1+t)^{-beta}$ for large abs $x$ and $t$, $a_0 > 0$, for $alpha$ belongs to $(-infinity, 1)$, $beta$ belongs to $(-1, 1)$. We established decay estimates for the energy, $L^2$ and $L^{p+1}$ norm of the solutions.

1. For $alpha$ belongs to $[0, 1)$, $beta$ belongs to $(-1, 1)$ and $alpha + beta$ belongs to $(0, 1)$, three different regimes of decay of solutions were found depending on the exponent of the absorption term, $p_1(n, alpha, beta) := 1 + 4(beta+1)/(2(n - alpha)(beta+1) - beta(2 - alpha))$ is a critical exponent in the following sense. For the supercritical region, namely $p$ belongs to $(p_1(n, alpha, beta), (n+2)/(n-2)$, the decay of solutions of the nonlinear equation coincides with the decay of the corresponding linear problem. For the subcritical region $p$ belongs to $(1, p_1(n, alpha, beta))$ the decay is much faster. Moreover, the subcritical region is divided into two subregions with completely different decay rates by another critical exponent $p_2 := 1 + (2alpha)/(n - alpha)$. If $p$ belongs to $(1, p_2(n, alpha, beta))$ the decay of solutions becomes independent of $alpha$ and $beta$.

2. For $alpha$ belongs to $(-infinity, 0)$ and $beta$ belongs to $(-1, 1)$. Two different regimes of decay of solutions were found depending on the exponent of the absorption term. $p_1(n, alpha, beta) := 1 + 4(1 - beta)/(2(n + alpha)(1 - beta) + beta(2 + alpha))$ is a critical exponent in the following sense. For the supercritical region, namely $p$ belongs

to $(p_1(n, alpha, beta), (n+2)/(n-2))$, the decay of solutions of the nonlinear equation coincides with the decay of the corresponding linear problem. For the subcritical region $p$ belongs to $(1, p_1(n, alpha, beta)$ the decay is much faster.

We study also the long time behavior of solutions of the wave equations with focusing - abs $(u(t,x))^{p-1}u(t,x)$ and variable damping $a(t,x)u_t(t,x)$, where $p$ belongs to $(1, n+2/n-2)$ and $a(t,x) \sim a_0(1+absx)^{-alpha}(1+t)^{-beta}$ for large abs $x$ and $t$, $a_0 > 0$, for $alpha$ belongs to $(0,1)$, $beta$ belongs to $(-1,1)$. A sharp critical exponents results were found depending on the exponent of the focusing term, for supercritical region, namely; $p$ belongs to $(p(n, alpha, beta) := 1 + 4(beta+1)/2(n-alpha)(beta+1) - beta(2-alpha), n-2/n+2)$ the solutions are global for all small data. We also established decay estimates for the energy, $L^2$ and $L^{p+1}$ norm of the solutions.

# Contents

# Chapter 1

# Introduction

## 1.1 Notations and Definitions

Throughout this thesis, we are using the following notations.

Let $x \in \mathbf{R}^n$ denote a vector $x = (x_1, x_2, \dots, x_n)$ and $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ denote its length. The radial and angular components of $x$ are $r = |x|$ and $\omega = x/|x|$, respectively. The dot product of two vectors is defined by $x \cdot y = x_1 y_1 + \cdots x_n y_n$.

We are using standard notations for the partial derivatives, gradient, and Laplace operator:

$$u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \quad \nabla u = \left(\frac{\partial u}{\partial x_i}\right)_{1 \le i \le n}, \quad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

We also write $D^\alpha u$, $\alpha \geq 0$, to denote the set of $\alpha^{th}$ order partial derivatives of $u$.

The Lebesgue space $L^p(\mathbf{R}^n)$, where $1 \le p \le \infty$, consists of equivalence classes of measurable functions on $\mathbf{R}^n$, such that the norm

$$\|u\|_{L^p} := \begin{cases} \left( \int_{\mathbf{R}^n} |u(x)|^p \, dx \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x \in \mathbf{R}^n} |u(x)|, & p = \infty, \end{cases}$$

is finite. For integer $m \geq 0$, we define the Sobolev spaces

$$W^{m,p}(\mathbf{R}^n) = \{u : \ D^\alpha u \in L^p(\mathbf{R}^n), 0 \leq \alpha \leq m\}$$

with the norms

$$\|u\|_{W^{m,p}} := \sum_{\alpha=0}^{m} \|D^\alpha u\|_{L^p}.$$

The $L^2$–Sobolev spaces $W^{m,2}(\mathbf{R}^n)$ are also denoted by $H^m(\mathbf{R}^n)$. We will study the functional of energy

$$E(u,t) = \frac{1}{2}\int_{\mathbf{R^n}} (u_t^2 + |\nabla u|^2)\ dx,$$

where $u$ will be a solution, in a weak sense, to a second-order dissipative wave equation on $(0,T^*) \times \mathbf{R}^n$. A natural domain of $E(\cdot,t)$ is the Banach space

$$X_1(0,T^*) := C([0,T^*); H^1(\mathbf{R}^n)) \cap C^1([0,T^*); L^2(\mathbf{R}^n)), \tag{1.1.1}$$

which is called the energy space. The initial data $(u,u_t)|_{t=0}$ will be functions from the product space $\mathcal{H}_0^1(\mathbf{R}^n)$ defined as

$$\mathcal{H}_0^1(\mathbf{R}^n) := \{(u_0,u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n) : \text{supp } (u_0,u_1) \text{ is compact}\}.$$

To recall the definitions of weak and classical solutions, consider the problem

$$u_{tt} - \Delta u + a(t,x)u_t = h(t,x), \quad x \in \mathbf{R}^n, \quad t > 0, \tag{1.1.2}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.1.3}$$

where $a(t,x) \in C^1(\mathbf{R} \times \mathbf{R}^n)$ and $h(t,x) \in L^1_{loc}(\mathbf{R} \times \mathbf{R}^n)$.

**Definition 1.1.1.** A weak solution of the Cauchy problem (1.1.2)–(1.1.3) is $u \in L^1_{loc}(\mathbf{R}\times\mathbf{R}^n)$, such that for all $\varphi \in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ the following holds:

$$\begin{aligned} \int_0^\infty \int_{\mathbf{R}^n} u(\varphi_{tt} - \Delta\varphi - (a\varphi)_t)\ dx\ dt &= \int_0^\infty \int_{\mathbf{R}^n} h\varphi\ dxdt + \int_{\mathbf{R}^n} a(0,\cdot)u_0\varphi(0,\cdot)\ dx \\ &+ \int_{\mathbf{R}^n} (u_0\partial_t\varphi(0,\cdot)\ dx - \int_{\mathbf{R}^n} u_1\varphi(0,\cdot)\ dx. \end{aligned}$$

A weak finite energy solution is a weak solution $u \in L^1_{loc}(\mathbf{R}\times\mathbf{R}^n)$, such that the additional condition $Du \in L^\infty_{loc}(\mathbf{R}; L^2_{loc}(\mathbf{R}^n))$ holds. If a weak solution $u$ satisfies an even stronger condition, $Du \in C(\mathbf{R}; L^2(\mathbf{R}^n))$, we simply call it a finite energy solution.
Where $D = (\nabla, \partial_t)$

**Definition 1.1.2.** A classical solution of the Cauchy problem (1.1.2)–(1.1.3) is a function $u \in C^2(\mathbf{R}\times\mathbf{R}^n)$ that satisfies the equation and assumes the initial data point wise. (It is clear that classical solutions are also weak solutions.)

The following classical result, for the Cauchy problem (1.1.2)–(1.1.3) holds (see [40]):

**Theorem 1.1.3.** *(Existence and uniqueness of weak solution). For the Cauchy problem (1.1.2)–(1.1.3), assume that $a(t,x) \in C^1(\mathbf{R}\times\mathbf{R}^n)$, $h(t,x) \in L^1_{loc}(\mathbf{R}\times\mathbf{R}^n)$ and $(u_0,u_1) \in H^1_0(\mathbf{R}^n)\times L^2(\mathbf{R}^n)$. There exists a unique weak finite energy solution $u$,*

$$Du \in L^\infty_{loc}(\mathbf{R}; L^2_{loc}(\mathbf{R}^n)).$$

In this thesis we consider nonlinear problems of the form

$$u_{tt} - \Delta u + a(t,x)u_t + k|u|^{p-1}u = 0, \quad x \in \mathbf{R}^n, \quad t > 0, \tag{1.1.4}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.1.5}$$

where $k \in \mathbf{R}$ and $p > 1$. A global result like Theorem 1.1.3 does not hold, in general, but local existence and uniqueness hold under additional assumptions.

**Theorem 1.1.4.** *(Local existence and uniqueness of solution). Assume that* $a(t,x) \in C^1(\mathbf{R} \times \mathbf{R}^n)$ *and* $(u_0, u_1)$ *are compactly supported functions,* $u_0 \in H_0^1(\mathbf{R}^n)$*, and* $u_1 \in L^2(\mathbf{R}^n)$*. Moreover, assume that* $1 < p < (n+2)/(n-2)$ *if* $n \geq 3$*, and* $1 < p < \infty$ *if* $n = 1, 2$*. There exists* $T^* > 0$*, then problem (1.1.4)–(1.1.5) admits a unique finite energy solution* $u$ *on* $(0, T^*)$*, with regularity*

$$Du \in C((0, T^*); L^2(\mathbf{R}^n)).$$

*Let* $B(R)$ *be a ball of radius* $R$*, and center at the origin. For the wave equation (1.1.4)–(1.1.5) the finite speed of propagation holds, namely*

$$\text{If supp } (u_0, u_1) \subset B(R) \text{ then supp } (u, u_t) \subset B(t+R), \text{ for any } 0 < t < T^*.$$

**Definition 1.1.5.** A global solution of problem (1.1.4)–(1.1.5) is a solution $u$, such that the conclusions of Theorem 1.1.4 hold for all $T^* > 0$.

There is a simple test for existence of global solutions called the continuation principle; it basically says that a solution can be continued as long as its energy remains finite.

**Theorem 1.1.6.** *Let* $u$ *be a finite energy solution of problem (1.1.4)–(1.1.5) on* $(0, T^*)$*. Then one of the following possibilities is realized:*

1. $T^* = \infty$; *u is a global solution.*

2. $T^* < \infty$ *and* $\lim_{t \to T^{*-}} (\|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2) = \infty$; *u blows up in finite time.*

An immediate consequence of this result is that the solution in Theorem 1.1.4 is global when $a(t,x) \geq 0$ and $k \geq 0$. Multiply the wave equation (1.1.4) with $u_t$ and integrate by part over $\mathbf{R}^n$, and using the finite speed of propagation to cancel the boundary integral we have:

$$E(u,t) = \frac{1}{2}\int_{\mathbf{R^n}} (u_t^2 + |\nabla u|^2)\ dx + \frac{k}{p+1}\int_{\mathbf{R^n}} |u|^{p+1}\ dx \leq E(u,0);$$

hence $\|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 < \infty$ at all $t \geq 0$. together with the continuation principle we have the following global existence and uniqueness theorem:

**Theorem 1.1.7.** *Assume that $a(t,x) \in C^1(\mathbf{R} \times \mathbf{R}^n)$ is a positive function, and $(u_0, u_1)$ are compactly supported functions, $u_0 \in H_0^1(\mathbf{R}^n)$, $u_1 \in L^2(\mathbf{R}^n)$, and $k > 0$. Moreover, assume that $1 < p < (n+2)/(n-2)$, if $n \geq 3$, and $1 < p < \infty$ for $n = 1, 2$. For any $T^* > 0$, the problem (1.1.4)–(1.1.5) admits a unique finite energy solution $u$ on $(0, T^*)$, with regularity*

$$Du \in C((0,\infty); L^2(\mathbf{R}^n)).$$

## 1.2 Useful Formulas and Inequalities

We provide a list of some frequently used inequalities.

**Young's Inequality**: Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a, b \geq 0.$$

**Hölder's Inequality**: Assume that $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\Omega \subset \mathbf{R}^n$. Then

$$\int_\Omega |uv|\ dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)},$$

for all $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$.

**Gagliardo–Nirenberg Inequality**: Let $1 < p < n$. There exists $C > 0$, such that

$$\|u\|_{L^r} \leq C \|u\|_{L^q}^{1-\theta} \|\nabla u\|_{L^p}^{\theta},$$

where

$$p < q \leq \frac{(n-1)p}{n-p}, \quad r = \frac{p(q-1)}{p-1}, \quad \theta = \frac{n(q-p)}{(np-(n-p)q)(q-1)}.$$

**Poincaré's inequality**: Let $1 < p < \infty$ and assume that $u = 0$ for $|x| > R$. Then

$$\|u\|_{L^p} \leq C_p R \|\nabla u\|_{L^p},$$

whenever $\|\nabla u\|_{L^p} < \infty$.

We often refer to the formula for integration in spherical coordinates.

**Theorem 1.2.1.** *(Spherical coordinates). Let $f : \mathbf{R}^n \to \mathbf{R}$ be continuous and integrable. Let $dS_r$ be the surface measure on the sphere of radius $r$. Then*

$$\int_{\mathbf{R}^n} f \; dx = \int_0^\infty \left( \int_{\partial B(x_0, r)} f \; dS_r \right) dr.$$

Below we assume that $\Omega$ is a bounded and open subset of $\mathbf{R}^n$ and its boundary $\partial\Omega$ is $C^1$. The following formulas are very useful.

**Theorem 1.2.2.** *(Integration-by-parts formula). Let $u, v \in C^1(\overline{\Omega})$. Then*

$$\int_\Omega u_{x_i} v \; dx = -\int_\Omega u v_{x_i} \; dx + \int_{\partial\Omega} u v \nu^i \; dS, \quad (i = 1, 2, \ldots, n).$$

**Theorem 1.2.3.** *(Green's formulas) Let $u, v \in C^2(\overline{\Omega})$. Then*

$$(i) \int_\Omega \Delta u \; dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \; dS,$$

$$(ii) \int_\Omega Dv \cdot Du \; dx = -\int_\Omega u \Delta v \; dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} u \; dS.$$

## 1.3 The History of the Problems

In this thesis we consider two types of nonlinear wave equations with space–time dependent damping: focusing and defocusing. The problem is not only to study the global existence or blow-up of some solutions, but also the behavior of global solutions as time $t \to \infty$. We mostly consider the latter part, where we derive sharp weighted estimates for the energy and $L^2$-norm. We thoroughly investigate the effects of both damping coefficient and nonlinear term on the asymptotic behavior of solutions. Our results lead to new critical exponents and significantly generalize [44], [45].

We begin with the linear dissipative wave with a space–time dependent potential

$$u_{tt} - \Delta u + a(t,x)u_t = 0, \quad x \in \mathbf{R}^n, \quad t > 0, \tag{1.3.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.3.2}$$

where $(u_0, u_1)$ are compactly supported initial data from the energy space $\mathcal{H}_0^1(\mathbf{R}^n)$. The potential $a(t,x)$ is a positive $C^1(\mathbf{R}_+, \mathbf{R^n})$ function. By Theorem (1.1.7) the problem (1.3.1)–(1.3.2) has a unique finite energy solution $u$, such that

$$u \in C((0,\infty), H^1(\mathbf{R}^n)), \quad u_t \in C((0,\infty), L^2(\mathbf{R}^n)).$$

Recently, there is an increasing interest in deriving sharp $L^p$ decay estimates. Before discussing current works, we should mention a few classical results. In a pioneering paper, Matsumura [25] studied (1.3.1) with constant damping coefficient $a(t,x) = a_0 > 0$, using Fourier analysis he showed the solution $u$ satisfies

$$\begin{aligned} \int_{\mathbf{R}^n} u^2 \, dx &\leq Ct^{-\frac{n}{2}}, \\ \int_{\mathbf{R}^n} (u_t^2 + |\nabla u|^2) \, dx &\leq Ct^{-\frac{n}{2}-1}, \end{aligned}$$

where $C$ depends on the initial data. Thus, the decay rates increase with the dimension $n$. Such estimates become much sharper than the estimates derived using the multiplier method as $n$ increases; the multiplier method usually gives dimension–independent estimate, such as $O(t^{-\frac{1}{2}})$ for all $n \geq 2$. The case where the potential $a(t,x)$ is variable is quite different, since it presents a serious difficulty to Fourier techniques. Here the multiplier method is remarkably effective and leads to

$$\int_{\mathbf{R}^n} (u_t^2 + |\nabla u|^2)\, dx \leq Ct^{-\min\{2,a_0\}},$$

for $a(t,x) \geq a_0(1+t+|x|)^{-1}$. This is shown by Matsumura [26] and Uesaka [46] using multipliers of the form $(w(t)u)_t$ with suitable weights $w(t)$.

The study of time–dependent coefficients $a(t,x) = a_0(1+t)^{\beta}$, $\beta \in (-1,1)$, has been initiated by Reissig [33]. Similarly to the case of constant coefficients, Fourier analysis is the most powerful technique for decay estimates. Wirth [48], [49], [50] and Reissig and Wirth [34] have consequently found sharp $L^p - L^q$ estimates for problem (1.3.1)–(1.3.2) including

$$\begin{aligned} \int_{\mathbf{R}^n} u^2\, dx &\leq Ct^{-(1-\beta)\frac{n}{2}}, \\ \int_{\mathbf{R}^n} (u_t^2 + |\nabla u|^2)\, dx &\leq Ct^{-(1-\beta)(\frac{n}{2}+1)}. \end{aligned}$$

An interesting observation is that the two decay rates increase as $\beta$ decreases from 1 to $-1$. Hence, moderately small coefficients dissipate energy more effectively than large coefficients. The decay rates are 0 when $\beta = 1$, which is known as over damping. When the damping is $a(x) \sim a_0(1+|x|)^{-\alpha}$, with $a_0 > 0$, and $\alpha \in [0,1)$, Fourier techniques are not applicable and classical multipliers are no longer suitable. This was first shown by Ikehata [11] who used an exponential multiplier to derive

$$\int_{\mathbf{R}^n} e^{2a_0(2-\alpha)^{-2}\frac{|x|^{2-\alpha}}{t}} (u_t^2 + |\nabla u|^2)\, dx \leq C$$

for large $t$. Such weights are typical for parabolic equations, but incompatible with conservative hyperbolic (wave, Klein-Gordon, etc) equations. Todorova and Yordanov [44] improved the multiplier method for the wave equations with damping. Their approach consists of four main steps:

1. It is possible to find an approximate solution of equation (1.3.1) which is a relatively simple function $w$ resembling the Gaussian kernel. Here the diffusion phenomenon plays a key role; this was discovered by Narazaki [31] for constant damping $a(t,x) = 1$. Then $u \sim Cw$ for a solution $w$ of the diffusion equation

$$w_t - \Delta w = 0, \quad x \in \mathbf{R}^n, \quad t > 0.$$

We can use the Gaussian

$$w(t,x) = t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$$

as an approximate solution of (1.3.1).

Wirth [48] noticed the same phenomenon when the coefficient is a function of time, i.e $a(t,x) = a(t)$. The approximate solution $w$ solves a parabolic equation:

$$a(t)w_t - \Delta w = 0, \quad x \in \mathbf{R}^n, \quad t > 0.$$

Todorova and Yordanov [44] conjectured that the phenomenon persists when the damping coefficient is $a(t,x) = a(x)$. The suggested approximate solution $w$ comes from

$$a(x)w_t - \Delta w = 0, \quad x \in \mathbf{R}^n, \quad t > 0.$$

Now we readily construct approximations of the form

$$w(t,x) = t^{-\frac{m}{2}} e^{-\frac{S(x)}{t}},$$

with a suitable function $S(x) \sim S_0(1+|x|)^{2-\alpha}$. Recall that we expect $u \sim Cw$. Once the asymptotic profile $w$ is found, we factor it out and work with $v = w^{-1}u$. The quotient $w^{-1}u$ varies slowly and admits more precise estimates.

2. Derive a modified equation for $v = w^{-1}u$ and a weighted energy identity for $v$ by multiplying the new equation with $w_0 v + w_1 v_t$, where $w_0$, $w_1$ are suitable weights.

3. Choose the weights so that the energy of $v$ also dissipates. This leads to several conditions on the weights $w_0$ and $w_1$.

4. Going back from $v$ to $u$, and obtain decay estimates for $u$.

Following these steps, Todorova and Yordanov [44] have found sharp decay estimates for the energy and $L^2$ norm of solutions to (1.3.1). If the potential $a(x)$ is a radially symmetric function $a(x) \in C^1(\mathbf{R}^n)$, such that $a(x) > 0$ and

$$a(x) \sim a_0(1+|x|)^{-\alpha}, \quad |x| \to \infty,$$

for $\alpha \in [0,1)$, and $a_0 > 0$ their results are

$$\begin{aligned} \int_{\mathbf{R}^n} e^{a_2(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} u^2 \, dx &\leq Ct^{\delta - \frac{n-2\alpha}{2-\alpha}}, \\ \int_{\mathbf{R}^n} e^{a_2(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} (u_t^2 + |\nabla u|^2) \, dx &\leq Ct^{\delta - \frac{n-\alpha}{2-\alpha} - 1}, \end{aligned}$$

where $t \geq 1$ and $\delta > 0$ can be any number; when $\alpha = 0$, the above results coincide with the exact results of Matsamura [25] found by Fourier analysis. The space–time dependent coefficients $a(t,x)$ present additional difficulties to the sharp version of multiplier method. The case of separable coefficients is studied in Kenigson [KK] under the assumptions that $a(t,x) = \lambda(x)\eta(t) \in C^1(\mathbf{R}_+, \mathbf{R}^n)$ is radially symmetric with respect to $x$ and satisfies $a(t,x) \sim a_0(1+|x|)^{-\alpha}(1+t)^{-\beta}$ for $\alpha \in [0,1)$ and $\beta \in (-1,1)$, such that $0 < \alpha + \beta < 1$.

These authors have found the following estimates for equation (1.3.1):

$$
\begin{aligned}
\int_{\mathbf{R}^n} e^{(\frac{n-\alpha}{2-\alpha}-2\delta)\frac{\phi_0|x|^{2-\alpha}}{t^{\beta+1}}} u^2\,dx &\leq Ct^{(\beta+1)(2\delta-\frac{n-2\alpha}{2-\alpha})+\beta}, \\
\int_{\mathbf{R}^n} e^{(\frac{n-\alpha}{2-\alpha}-2\delta)\frac{\phi_0|x|^{2-\alpha}}{t^{\beta+1}}} (u_t^2+|\nabla u|^2)\,dx &\leq Ct^{(\beta+1)(2\delta-\frac{n-\alpha}{2-\alpha})-1}.
\end{aligned}
$$

We are going to define some terms and expression as presented in Todorova and Yordanov [44], which will be included in our theorems.

Under the assumptions that $a(t,x) = \lambda(x)\eta(t) \in C^1(\mathbf{R}_+, \mathbf{R}^n)$ is radially symmetric with respect to $x$ and satisfies $a(t,x) \sim a_0(1+|x|)^{-\alpha}(1+t)^{-\beta}$ for $\alpha \in (-\infty, 1)$ and $\beta \in (-1,1)$. Assume that the Poisson equation

$$\Delta\phi(x) = \lambda(x), \quad x \in \mathbf{R}^n. \tag{1.3.3}$$

has a nonnegative solution $\phi(x)$ with the following properties

$$
\begin{aligned}
&(a1) \qquad \phi(x) \geq 0 \ \text{ for all } \ x \in \mathbf{R}^n, \\
&(a2) \qquad \phi(x) = O(|x|^{2-\alpha}),\ \alpha \in [0,1) \text{ for large } |x|, \\
&\qquad\qquad \phi(x) = O(|x|^{2+\alpha}),\ \alpha \in (-\infty,0) \text{ for large } |x|, \\
&(a3) \qquad m(\lambda) = \liminf_{x\to\infty} \frac{\lambda(x)\phi(x)}{|\nabla\phi(x)|^2} \ > \ 0.
\end{aligned}
$$

Generally speaking there are many functions, satisfying $(a1)$–$(a3)$, including a radially symmetric one.

A special case is where $\lambda(x) > 0$ is a radially symmetric $C^1$ function, such that for $\alpha \in [0,1)$,

$$\lambda(x) \sim \lambda_2|x|^{-\alpha} \text{ as } |x| \to \infty, \tag{1.3.4}$$

with $\lambda_2 > 0$. As shown in [44] the Poisson equation (3.1.5) has a solution $\phi(x)$ that satisfies:

$$\phi(x) \sim \frac{\lambda_2}{(2-\alpha)(n-\alpha)}|x|^{2-\alpha}, \ |x| \to \infty.$$

In this case the rate $m(\lambda)$ at which the solution decays, depends on the space dimension and the decay rate $\alpha$, and can be given as

$$m(\lambda) = \frac{n-\alpha}{2-\alpha}.$$

While for $\alpha \in (0, \infty)$,

$$\lambda(x) \sim \lambda_2 |x|^{\alpha} \text{ as } |x| \to \infty, \tag{1.3.5}$$

with $\lambda_2 > 0$. It is shown in Appendix ($C$) that the Poisson equation (3.1.5) has a solution $\phi(x)$ that satisfies:

$$\phi(x) \sim \frac{\lambda_2}{(2+\alpha)(n+\alpha)}|x|^{2+\alpha}, \ |x| \to \infty.$$

In this case the rate $m(\lambda)$ at which the solution decays, depends on the space dimension and the decay rate $\alpha$, and can be given as

$$m(\lambda) = \frac{n+\alpha}{2+\alpha}.$$

**The first goal** of this thesis is to generalize the aforementioned results to a wider class of damping coefficients. Thus, we study the asymptotic behavior of solutions of the dissipative wave equation (1.3.1)–(1.3.2) with space–time dependent potentials of the form $a(t,x) = \lambda(x)\eta(t)$, where $\lambda(x)$ is radially symmetric function of $x$. It is important to mention that we allow damping coefficients $a(t,x)$ go to infinity as $t$ or $|x| \to \infty$. In fact, the potential $a(t,x) = \lambda(x)\eta(t)$ is positive $C^1(\mathbf{R}_+ \times \mathbf{R}^n)$, where $\lambda(x)$ and $\eta(t)$ satisfy the following conditions:

$$\lambda_0(1+|x|)^\alpha \leq \quad \lambda(x) \quad \leq \lambda_1(1+|x|)^\alpha, \quad \alpha \in (0,\infty), \tag{1.3.6}$$

$$\eta_0(1+t)^\beta \leq \quad \eta(t) \quad \leq \eta_1(1+t)^\beta, \quad \beta \in (-1,1). \tag{1.3.7}$$

for all $(t,x) \in (\mathbf{R}_+, \mathbf{R}^n)$, $\lambda_0$, $\lambda_1$, $\eta_0$, $\eta_1 > 0$, such that

$$\frac{\eta_0^2}{\eta_1^2(1-\beta)^2} > \frac{\lambda_1}{3\lambda_0(n+\alpha)(2+\alpha)}. \tag{1.3.8}$$

Our technique modifies and improves [44].

Below we summarize the main decay estimates.

**Theorem 1.3.1.** *Let the potential $a(t,x) = \lambda(x)\eta(t)$ be $C^1(\mathbf{R}_+, \mathbf{R^n})$ function which is radially symmetric with respect to $x$, such that conditions (1.3.6), (1.3.7) and (1.3.8) satisfy. Suppose that conditions (a1)–(a3) hold. Then the solution of*

$$u_{tt} - \Delta u + a(t,x)u_t = 0, \quad x \in \mathbf{R}^n, \quad t > 0, \tag{1.3.9}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.3.10}$$

*satisfies the following decay estimates*

$$\begin{aligned} \int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} u^2 \, dx &\leq C(1+t)^{-\beta+(1-\beta)(2\delta-m(\lambda))} \\ \int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} (u_t^2 + |\nabla u|^2) dx &\leq C(1+t)^{(1-\beta)(2\delta-m(\lambda))-1} \end{aligned}$$

*for all $t > t_0$, where $m(\lambda)$ is given in (a3) and $\delta > 0$ is arbitrary small number. The constant $C$ depends on the radius of the support $R$, $a$, $\delta$, and $n$.*

An important consequence of the main theorem is that all norms under consideration, restricted to $\{\phi(x) \geq T^{1+\epsilon}\}$ with $\epsilon > 0$, decay exponentially.

**Corollary 1.3.2.** *Let the potential $a(t,x) = \lambda(x)\eta(t)$ be $C^1(\mathbf{R}_+, \mathbf{R^n})$ function which is radially symmetric with respect to $x$, such that conditions (1.3.6), (1.3.7) and (1.3.8) satisfy. Suppose also that conditions (a1)–(a3) hold. Then for $\delta > 0$ arbitrary small number and $\epsilon > 0$ the solution of (1.3.9)–(1.3.10) satisfies*

$$\int_{\phi(x)\geq(1+t)^{(1+\epsilon)(1-\beta)}} (u^2 + u_t^2 + |\nabla u|^2)\, dx \quad \leq \quad Ce^{-(m(\lambda)-2\delta)(1+t)^{\epsilon(1-\beta)}},$$

*where $t > t_0$. The constant $C$ depends on the radius of the support $R$, $a$, $\delta$, and $n$.*

**Corollary 1.3.3.** *Assume that the potential $a(t,x) = \lambda(x)\eta(t)$ is a $C^1(\mathbf{R}_+, \mathbf{R^n})$ function which is radially symmetric with respect to $x$, such that conditions (1.3.5), (1.3.6), (1.3.7) and (1.3.8) satisfy. Then for $\delta > 0$ arbitrary small number the solution of (1.3.9)–(1.3.10) satisfies*

$$\int_{\mathbf{R}^n} e^{c_0(\frac{n+\alpha}{2+\alpha}-2\delta)\frac{|x|^{2+\alpha}}{(1+t)^{1-\beta}}} u^2\, dx \quad \leq \quad C(1+t)^{-\beta+(1-\beta)(2\delta-\frac{n+\alpha}{2+\alpha})},$$

$$\int_{\mathbf{R}^n} e^{c_0(\frac{n+\alpha}{2+\alpha}-2\delta)\frac{|x|^{2+\alpha}}{(1+t)^{1-\beta}}} (u_t^2 + |\nabla u|^2)\, dx \quad \leq \quad C(1+t)^{(1-\beta)(2\delta-\frac{n+\alpha}{2+\alpha})-1},$$

*where $t > t_0$, $c_0$ depends on $\phi_0$ and $\beta$. The constant $C$ depends on the radius of the support $R$, $a$, $\delta$, $\phi_0$, and $n$.*

In this thesis **we also studied the dissipative wave equations** with nonlinear absorption. Our main example is the following Cauchy problem:

$$u_{tt} - \Delta u + a(t,x)u_t + |u|^{p-1}u = 0, \quad x \in \mathbf{R}^n, \quad t > 0, \tag{1.3.11}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.3.12}$$

The initial data $(u_0, u_1) \in \mathcal{H}_0^1(\mathbf{R}^n)$, i.e. these are compactly supported functions and

$$u_0 \in H^1(\mathbf{R}^n), \quad u_1 \in L^2(\mathbf{R}^n).$$

We consider the usual potentials $a(t,x) \in C^1(\mathbf{R}_+ \times \mathbf{R}^n)$ and exponents $p$ satisfying $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$, and $1 < p < \infty$ for $n = 1, 2$.

The global well-possedness of (1.3.11)–(1.3.12) is a classical result of Strauss [40]. He showed that $E(u,t) \leq E(u,0)$ for $t \geq 0$ and

$$u \in C((0,\infty), H^1(\mathbf{R}^n)), \quad u_t \in C((0,\infty), L^2(\mathbf{R}^n))$$

for any data in the energy space. Hence, it is natural to study the asymptotic behavior of $E(u,t)$ and other norms as $t \to \infty$.

The most accessible case of problem (1.3.11)–(1.3.12) is when $a(t,x) = 1$. Decay estimates for the solution $u$ have already been obtained. The diffusion phenomenon insures that, for large $t$, $u$ will be similar to the corresponding solution of the heat equation

$$u_t - \Delta u + |u|^{p-1}u = 0, \quad x \in \mathbf{R}^n, \quad t > 0.$$

This allowed Kawashima, Nakao, and Ono [21] to treat the case of low dimensions and relatively high exponents of the absorption term: $1 + \frac{4}{n} < p \leq \frac{n+2}{n-2}$ for $n = 3$ and $p > 1 + \frac{4}{n}$ for $n = 1, 2$. They combined $L^p - L^q$ decay estimates for the linear problem with energy estimates to show that

$$\|u\|_{L^2} = O(t^{-\frac{n}{2}(\frac{1}{r} - \frac{1}{2})})$$

where the initial data $(u_0, u_t) \in (H^1 \cap L^r)(\mathbf{R}^n) \times (L^2 \cap L^r)(\mathbf{R}^n)$ with $(1 \leq r \leq 2)$.

A closely related equation is

$$u_{tt} - \Delta u + u_t = |u|^p, \quad x \in \mathbf{R}^n, \quad t > 0,$$

which is called the focusing nonlinear wave equation with damping. Todorova and Yordanov [43] have shown that the supercritical case $p > 1 + \frac{2}{n}$ with small data admits global solutions

$u$, such that

$$\|Du\|_{L^2} = O(t^{-\frac{n}{4}-\frac{1}{2}}).$$

The latter decay rate is identical with the decay rate of the linear heat equation. Karch [19] has developed the approach of [21] in order to find not only the decay rate but also the leading term in the asymptotic of $u$ as $t \to \infty$. He showed that, for $p > 1 + \frac{4}{n}$ and $1 \leq n \leq 3$, the asymptotic profiles of solutions resemble the Gauss kernel. Recently Hayashi, Kaikina, Naumkin [7] extended this result to all supercritical $p > 1 + \frac{2}{n}$ when $n = 1$. The decay rates and asymptotic profiles are expected to be quite different in the subcritical case $1 < p < 1 + \frac{2}{n}$, although similarities with the nonlinear heat equation $u_t - \Delta u + |u|^{p-1}u = 0$ are expected to persist. Nishihara and Zhao [32] studied problem (1.3.11)–(1.3.12) when $n = 2$ and $(u_0, u_1)$ decay exponentially as $x \to \infty$. Using the multiplier method with weights $e^{\psi(t,x)}$, where $\psi(t,x) = \frac{a|x|^2}{4(t+t_0)}$, $0 < a < 1$, $t_0 \geq 1$, they established the estimates

$$(\|u\|_{L^{p+1}}, \|\nabla u\|_{L^2}) = O(t^{-\frac{1}{p-1}+\frac{n}{2(p+1)}}, t^{-\frac{1}{p-1}+\frac{n}{4}-\frac{1}{2}}).$$

Interestingly the decay rates depend on the exponent of nonlinearity $p$, similar to the decay rates of Escobedo and Kavian [1] for the corresponding norms of the nonlinear heat equation. Thus, there has been a gap between subcritical $p$ and large $p$, described by $1+\frac{2}{n} < p < 1+\frac{4}{n}$, in which decay estimate have been more difficult to obtain. Important partial results have been given by Ikehata, Nishihara, and Zhao [15] for $n = 3$ and small data or $n = 1, 2$ and arbitrarily large data:

$$\|u\|_{L^2} \leq C(1+t)^{-\frac{1}{p-1}+\frac{n}{4}},$$

$$\|u_t\|_{L^2} + \|\nabla u\|_{L^2} \leq C(1+t)^{-\frac{1}{p-1}-\frac{1}{2}+\frac{n}{4}}.$$

The method of [15] does not work in higher dimensions $n \geq 3$ even for $a(t,x) = 1$. It turns out that a strengthened version of the multiplier method can yield sharp decay estimates for any dimension and size of initial data, in both subcritical and supercritical cases. The results of Todorova and Yordanov [45] actually apply to space dependent potentials $a(t,x) = a(x)$, where $a(x) \in C^1(\mathbf{R^n})$ is a radially symmetric function of $|x|$ and

$$a(x) \sim a_0(1+|x|)^{-\alpha}, \quad a_0 > 0, \quad \alpha \in [0,1).$$

To describe some of their estimates, we assume that $n \geq 3$, $1 < p < \frac{n+2}{n-2}$, and $(u_0, u_1)$ are compactly supported functions from the energy space. There are three different regions of asymptotic behavior determined by the exponent of absorption term $p$ and the exponent of damping coefficient $\alpha$. The first threshold is $p_1(n,\alpha) = 1+\frac{2}{n-\alpha}$. For the supercritical values of $p$, namely $p_1(n,\alpha) < p < \frac{n+2}{n-2}$, the decay rates of problem (1.3.11)–(1.3.12) coincide with the decay rates of the corresponding linear problem. For the subcritical region, $1 < p < p_1(n,\alpha)$, the decay rates are much faster. Moreover, the subcritical region is further divided into two subregions with two different decay rates by the inner threshold $p_2(n,\alpha) = 1 + \frac{2\alpha}{n-\alpha}$. If $1 < p < p_2(n,\alpha)$, the decay of solutions can be close to exponential as the absorption problem is close to a Klein-Gordon problem. If $p_2(n,\alpha) < p < p_1(n,\alpha)$, the decay of solutions is intermediate. Notice that $p_2(n,0) = 1$, so only one threshold $p_c(n) = 1+\frac{2}{n}$ exists when $\alpha = 0$. We will extend the results of [45] concerning (1.3.11)–(1.3.12) to space–time dependent potentials

$$a(t,x) \sim a_0(1+|x|)^{-\alpha}(1+t)^{-\beta}, \quad a_0 > 0, \quad \alpha \in (-\infty,1), \quad \beta \in (-1,1).$$

Meanwhile we will substantially modify the technique of Todorova and Yordanov [44], [45]. Let us summarize the decay estimates for the energy, $L^2$ and $L^{p+1}$ norms. Here we make the usual assumptions that $1 < p < \frac{n+2}{n-2}$ and $(u_0, u_1) \in \mathcal{H}_0^1(\mathbf{R}^n)$.

**For the first case** where the potential $a(t,x) = \lambda(x)\eta(t)$ where $\lambda(x)$, $\eta(t)$ are given by

$$\lambda_0(1+|x|)^{-\alpha} \leq \lambda(x) \leq \lambda_1(1+|x|)^{-\alpha}, \quad \alpha \in [0,1), \tag{1.3.13}$$

$$\eta_0(1+t)^{-\beta} \leq \eta(t) \leq \eta_1(1+t)^{-\beta}, \quad \beta \in (-1,1); \tag{1.3.14}$$

For all $(t,x) \in (\mathbf{R}_+, \mathbf{R}^n)$, $\lambda_0$, $\lambda_1$, $\eta_0$, $\eta_1 > 0$, such that $0 < \alpha + \beta < 1$.

The following decay estimates are optimal for large exponent $p$.

**Theorem 1.3.4.** *Let $1 < p < (n+2)/(n-2)$ and assume the potential $a(t,x) = \lambda(x)\eta(t)$, with $\lambda(x)$ and $\eta(t)$ are defined by (1.3.13) and (1.3.14) respectively. Then the solution of (1.3.11) satisfies*

$$\begin{aligned}
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} u^2 \, dx &\leq Ct^{\beta+(\beta+1)(2\delta-m(\lambda)+\frac{\alpha}{2-\alpha})}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} (u_t^2 + |\nabla u|^2) \, dx &\leq Ct^{(\beta+1)(2\delta-m(\lambda))-1}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} |u|^{p+1} \, dx &\leq Ct^{(\beta+1)(2\delta-m(\lambda))-1},
\end{aligned}$$

*where $\delta > 0$ is an arbitrary small number and $t \geq 1$. The constant $C$ depends on $\delta$ and the initial data $u_0$ and $u_1$.*

When the exponent $p$ is close to 1 the decay estimates are as follows:

**Theorem 1.3.5.** *Let $1 < p < (n+2)/(n-2)$ and assume the potential $a(t,x) = \lambda(x)\eta(t)$, with $\lambda(x)$ and $\eta(t)$ are defined by (1.3.13) and (1.3.14) respectively. Then the solution of (1.3.11) satisfies*

$$\begin{aligned}
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} u^2 \, dx &\leq Ct^{\delta+(\beta+1)\left(1+\frac{\alpha}{2-\alpha}-\frac{p+1}{p-1}-\min\left\{\frac{p+1}{p-1}\frac{\alpha}{2-\alpha}-\frac{n}{2-\alpha},0\right\}\right)}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} (u_t^2 + |\nabla u|^2) \, dx &\leq Ct^{\delta-(\beta+1)\left(\frac{p+1}{p-1}+\min\left\{\frac{p+1}{p-1}\frac{\alpha}{2-\alpha}-\frac{n}{2-\alpha},0\right\}\right)}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} |u|^{p+1} \, dx &\leq Ct^{\delta-(\beta+1)\left(\frac{p+1}{p-1}+\min\left\{\frac{p+1}{p-1}\frac{\alpha}{2-\alpha}-\frac{n}{2-\alpha},0\right\}\right)},
\end{aligned}$$

*where $\delta > 0$ is an arbitrary small number and $t \geq 1$. The constant $C$ depends on $\delta$ and on the initial data $u_0$ and $u_1$.*

An interesting observation is that if we add the estimates of Theorem (1.3.4) or (1.3.5), and consider the region $\phi(x) \geq T^{1+\epsilon}$, $\epsilon > 0$ the decay estimates will be exponential. This shows the parabolic asymptotic profile of the solution of the problem (1.3.11)–(1.3.12).

**Corollary 1.3.6.** *Let $1 < p < (n+2)/(n-2)$ and assume the potential $a(t,x) = \lambda(x)\eta(t)$, with $\lambda(x)$ and $\eta(t)$ are defined by (1.3.13) and (1.3.14) respectively. Then the solution of (1.3.11)–(1.3.12) satisfies the estimate*

$$\int_{\phi(x)\geq t^{(1+\epsilon)(1+\beta)}} (u^2 + u_t^2 + |\nabla u|^2 + |u|^{p+1})\ dx \quad \leq \quad C e^{-(m(\lambda)-2\delta)t^{\epsilon(\beta+1)}}.$$

*where $\delta > 0$ is an arbitrary small number and $t \geq 1$. The constant $C$ is a positive constant which depends on $\alpha$, $p$, and $n$.*

For the potential $a(t,x) = \lambda(x)\eta(t)$, such that the space dependent part

$$\lambda(x) \sim \lambda_2 |x|^{-\alpha}, \quad |x| \to \infty,$$

with $\lambda_2 > 0$, is a positive radially symmetric $C^1$ function, the computed decay rate is $m(\lambda) = \frac{n-\alpha}{2-\alpha}$. To find the threshold $p_1(n,\alpha,\beta)$ we substitute $\frac{n-\alpha}{2-\alpha}$ for $m(\lambda)$ in the $L^2$ estimates for the solution in Theorem (1.3.4), then we set it equal to the decay estimates for the $L^2$ norm for the solution in Theorem (1.3.5) and solve for $p$. Hence the following explicit estimates can be given:

**Corollary 1.3.7.** *Let the potential $a(t,x) = \lambda(x)\eta(t)$ where $\lambda(x)$ and $\eta(t)$ are given by (1.3.13) and (1.3.14) respectively. Assume that*

$$p_1(n,\alpha,\beta) := 1 + \frac{4(\beta+1)}{2(n-\alpha)(\beta+1) - \beta(2-\alpha)} \leq p < \frac{n+2}{n-2}.$$

*Then the solution of (1.3.11)–(1.3.12) satisfies the following estimates:*

$$\begin{aligned}
\int_{\mathbf{R}^n} e^{c_0(\frac{n-\alpha}{2-\alpha}-2\delta)\frac{|x|^{2-\alpha}}{t^{\beta+1}}} u^2\, dx &\le Ct^{\beta+(\beta+1)(2\delta+\frac{2\alpha-n}{2-\alpha})},\\
\int_{\mathbf{R}^n} e^{c_0(\frac{n-\alpha}{2-\alpha}-2\delta)\frac{|x|^{2-\alpha}}{t^{\beta+1}}} (u_t^2+|\nabla u|^2)\, dx &\le Ct^{(\beta+1)(2\delta-\frac{n-\alpha}{2-\alpha})-1},\\
\int_{\mathbf{R}^n} e^{c_0(\frac{n-\alpha}{2-\alpha}-2\delta)\frac{|x|^{2-\alpha}}{t^{\beta+1}}} |u|^{p+1}\, dx &\le Ct^{(\beta+1)(2\delta-\frac{n-\alpha}{2-\alpha})-1},
\end{aligned}$$

*where* $t \ge 1$. *Here* $\delta > 0$ *is an arbitrary small number,* $c_0$ *depends on* $\phi_0$ *and* $\beta$. *The constant* $C$ *depends on* $\delta$ *and on the initial data* $u_0$ *and* $u_1$.

Notice that Corollary (1.3.7) shows that the solution of nonlinear problem (1.3.11)–(1.3.12) in the supercritical region $p_1(n,\alpha,\beta) \le p \le \frac{n+2}{n-2}$ coincides with the decay of the solution of the corresponding linear problem, see [20].

For the subcritical region $1 < p < p_1(n,\alpha,\beta)$, Theorem (1.3.5) shows that the decay estimates is faster and given by the following Corollary.

**Corollary 1.3.8.** *Let the potential* $a(t,x) = \lambda(x)\eta(t)$ *where* $\lambda(x)$ *and* $\eta(t)$ *are given by (1.3.13) and (1.3.14) respectively. Assume that*

$$1 < p < p_1(n,\alpha,\beta).$$

*Then the solution of (1.3.11)–(1.3.12) satisfies the following estimates:*

$$\begin{aligned}
\int_{\mathbf{R}^n} e^{c_0(\frac{n-\alpha}{2-\alpha}-2\delta)\frac{|x|^{2-\alpha}}{t^{\beta+1}}} u^2\, dx &\le Ct^{\delta+(\beta+1)\left(1+\frac{\alpha}{2-\alpha}-\frac{p+1}{p-1}-\min\left\{\frac{p+1}{p-1}\frac{\alpha}{2-\alpha}-\frac{n}{2-\alpha},0\right\}\right)},\\
\int_{\mathbf{R}^n} e^{c_0(\frac{n-\alpha}{2-\alpha}-2\delta)\frac{|x|^{2-\alpha}}{t\beta+1}} (u_t^2+|\nabla u|^2)\, dx &\le Ct^{\delta-(\beta+1)\left(\frac{p+1}{p-1}+\min\left\{\frac{p+1}{p-1}\frac{\alpha}{2-\alpha}-\frac{n}{2-\alpha},0\right\}\right)},\\
\int_{\mathbf{R}^n} e^{c_0(\frac{n-\alpha}{2-\alpha}-2\delta)\frac{|x|^{2-\alpha}}{t^{\beta+1}}} |u|^{p+1}\, dx &\le Ct^{\delta-(\beta+1)\left(\frac{p+1}{p-1}+\min\left\{\frac{p+1}{p-1}\frac{\alpha}{2-\alpha}-\frac{n}{2-\alpha},0\right\}\right)},
\end{aligned}$$

*where* $\delta > 0$ *is an arbitrary small number and* $t \ge 1$. *Here* $c_0$ *depends on* $\phi_0$ *and* $\beta$. *The constant* $C$ *depends on* $\delta$ *and on the data* $u_0$ *and* $u_1$.

*Remark* 1.3.9. The subcritical region $1 < p < p_1(n, \alpha, \beta)$ is divided into two subregions with completely different decay rates as follows. For exponent $p$ such that $1 < p < 1 + \frac{2\alpha}{n-\alpha} =: p_2(n, \alpha)$ the decay rates are

$$(\|u\|_{L^{p+1}}, \|\nabla u\|_{L^2} + \|u_t\|_{L^2}) = O(t^{\delta-(\beta+1)\frac{1}{p-1}}, t^{\delta-\frac{\beta+1}{2}\frac{p+1}{p-1}}),$$

where $\delta$ is an arbitrary small number. In this region the decay rate is independent of $\alpha$. For the second subcritical region, namely the region of medium exponents $p_2(n, \alpha) < p < p_1(n, \alpha, \beta)$, the decay rate is

$$(\|u\|_{L^{p+1}}, \|\nabla u\|_{L^2} + \|u_t\|_{L^2}) = O(t^{\delta-(\beta+1)\left(\frac{1}{p-1}+\frac{\alpha}{(2-\alpha)(p-1)}-\frac{n}{(2-\alpha)(p+1)}\right)}, t^{\delta-\frac{\beta+1}{2}\left(\frac{p+1}{p-1}+\frac{p+1}{p-1}\frac{\alpha}{2-\alpha}-\frac{n}{2-\alpha}\right)}),$$

where $\delta$ is an arbitrary small number.

*Remark* 1.3.10. Let us mention that in the case of constant potential in (1.3.11) ($\alpha = 0$ and $\beta = 0$) the second critical exponent $p_2(n, \alpha) = 1 + \frac{2\alpha}{n-\alpha}$ is equal to 1 which explains why in the case of constant potential there are only two different regions with respect to the decay rate of the solution of (1.3.11). The first critical exponent $p_1(n, \alpha, \beta) = 1 + \frac{4(\beta+1)}{2(n-\alpha)(\beta+1)-\beta(2-\alpha)}$ becomes the Fujita critical exponent $1 + \frac{2}{n}$.

*Remark* 1.3.11. For the case where the potential is a space dependent $a(t, x) = a(x)$ in (1.3.11) ($\beta = 0$) our results coincides with Todorova and Yordanov [45].

**For the second case** where the potential $a(t, x) = \lambda(x)\eta(t)$ where $\lambda(x)$, $\eta(t)$ are given by

$$\begin{aligned} \lambda_0(1 + |x|)^\alpha \leq \quad & \lambda(x) \quad \leq \lambda_1(1 + |x|)^\alpha, \quad \alpha \in (0, \infty), && (1.3.15) \\ \eta_0(1 + t)^\beta \leq \quad & \eta(t) \quad \leq \eta_1(1 + t)^\beta, \quad \beta \in (-1, 1), && (1.3.16) \end{aligned}$$

for all $(t,x) \in (\mathbf{R}_+, \mathbf{R}^n)$, $\lambda_0$, $\lambda_1$, $\eta_0$, $\eta_1 > 0$, such that

$$\frac{\eta_0^2}{\eta_1^2(1-\beta)^2} > \frac{\lambda_1}{3\lambda_0(n+\alpha)(2+\alpha)}. \tag{1.3.17}$$

The following decay estimates are optimal for large exponents $p$.

**Theorem 1.3.12.** *Let $1 < p < (n+2)/(n-2)$ and assume that $a(t,x) = \lambda(x)\eta(t)$, where $\lambda(x)$, $\eta(t)$ are defined in (1.3.15), (1.3.16) such that condition (1.3.17) satisfies. Then the solution of (1.3.11)–(1.3.12) satisfies*

$$\begin{aligned}
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} u^2 \, dx &\le C(1+t)^{-\beta+(1-\beta)(2\delta-m(\lambda))}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} (u_t^2 + |\nabla u|^2) \, dx &\le C(1+t)^{(1-\beta)(2\delta-m(\lambda))-1}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} |u|^{p+1} \, dx &\le C(1+t)^{(1-\beta)(2\delta-m(\lambda))-1},
\end{aligned}$$

*where $\delta > 0$ is an arbitrary small number and $t \ge 1$. The constant $C$ depends on $\delta$ and the initial data $u_0$ and $u_1$.*

When the exponent $p$ is close to 1 the decay estimates are as follows:

**Theorem 1.3.13.** *Let $1 < p < (n+2)/(n-2)$ and assume that $a(t,x) = \lambda(x)\eta(t)$ where $\lambda(x)$, $\eta(t)$ are defined in (1.3.15), (1.3.16) such that condition (1.3.17) satisfies. Then the solution of (1.3.11)–(1.3.12) satisfies*

$$\begin{aligned}
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} u^2 \, dx &\le C(1+t)^{(1-\beta)\left(1-\frac{p+1}{p-1}+\frac{p+1}{p-1}\frac{\alpha}{2+\alpha}+\frac{n}{2+\alpha}\right)+\delta}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} (u_t^2 + |\nabla u|^2) \, dx &\le C(1+t)^{(\beta-1)\left(\frac{p+1}{p-1}-\frac{p+1}{p-1}\frac{\alpha}{2+\alpha}-\frac{n}{2+\alpha}\right)+\delta}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} |u|^{p+1} \, dx &\le C(1+t)^{(\beta-1)\left(\frac{p+1}{p-1}-\frac{p+1}{p-1}\frac{\alpha}{2+\alpha}-\frac{n}{2+\alpha}\right)+\delta},
\end{aligned}$$

*where $\delta > 0$ is an arbitrary small number and $t \ge 1$. The constant $C$ depends on $\delta$ and on the initial data $u_0$ and $u_1$.*

An important observation for both theorems is the following: the average decay rate of all norms under consideration in the region $\phi(x) \geq T^{1+\epsilon}$, $\epsilon > 0$, is exponential. In fact we can add the three estimates in Theorem (1.3.12) or Theorem (1.3.13) to obtain the following.

**Corollary 1.3.14.** *Let $1 < p < (n+2)/(n-2)$ and assume that $a(t,x) = \lambda(x)\eta(t)$ where $\lambda(x)$, $\eta(t)$ are defined in (1.3.15), (1.3.16) such that condition (1.3.17) satisfies. Then the solution of (1.3.11)–(1.3.12) satisfies the estimate*

$$\int_{\phi(x)\geq t^{(1+\epsilon)(1-\beta)}} (u^2 + u_t^2 + |\nabla u|^2 + |u|^{p+1})\ dx \quad \leq \quad Ce^{-(m(\lambda)-2\delta)t^{\epsilon(1-\beta)}}.$$

*where $C$ is a positive constant which depends on $\alpha$, $p$, and $n$.*

For the potential $a(t,x) = \lambda(x)\eta(t)$, such that the space dependent part

$$\lambda(x) \sim \lambda_2|x|^\alpha, \quad |x| \to \infty,$$

with $\alpha \in (0,\infty)$ and $\lambda_2 > 0$ is a positive radially symmetric $C^1$ function, the computed decay rate is $m(\lambda) = \frac{n+\alpha}{2+\alpha}$. To find the threshold $p_1(n,\alpha,\beta)$ we substitute $\frac{n+\alpha}{2+\alpha}$ for $m(\lambda)$ in the $L^2$ estimates for the solution in Theorem (1.3.12), then we set it equal to the decay estimates for the $L^2$ norm for the solution in Theorem (1.3.13) and solve for $p$. Hence the following explicit estimates can be given:

**Corollary 1.3.15.** *Let the potential $a(t,x) = \lambda(x)\eta(t)$ where $\lambda(x)$, $\eta(t)$ are defined in (1.3.15), (1.3.16) such that condition (1.3.17) satisfies. Assume that*

$$p_1(n,\alpha,\beta) := 1 + \frac{4(1-\beta)}{2(n+\alpha)(1-\beta)+\beta(2+\alpha)} \leq p < \frac{n+2}{n-2}.$$

*Then the solution of (1.3.11)–(1.3.12) satisfies the following estimates:*

$$\begin{aligned}\int_{\mathbf{R}^n} e^{c_0(\frac{n+\alpha}{2+\alpha}-2\delta)\frac{|x|^{2+\alpha}}{(1+t)^{1-\beta}}} u^2\, dx &\le C(1+t)^{-\beta+(1-\beta)(2\delta-\frac{n+\alpha}{2+\alpha})},\\ \int_{\mathbf{R}^n} e^{c_0(\frac{n+\alpha}{2+\alpha}-2\delta)\frac{|x|^{2+\alpha}}{(1+t)^{1-\beta}}} (u_t^2+|\nabla u|^2)\, dx &\le C(1+t)^{(1-\beta)(2\delta-\frac{n+\alpha}{2+\alpha})-1},\\ \int_{\mathbf{R}^n} e^{c_0(\frac{n+\alpha}{2+\alpha}-2\delta)\frac{|x|^{2+\alpha}}{(1+t)^{1-\beta}}} |u|^{p+1}\, dx &\le C(1+t)^{(1-\beta)(2\delta-\frac{n+\alpha}{2+\alpha})-1},\end{aligned}$$

*where $t \ge 1$. Here $\delta > 0$ is an arbitrary small number, $c_0$ depends on $\phi_0$ and $\beta$. The constant $C$ depends on $\delta$ and on the initial data $u_0$ and $u_1$.*

Corollary (1.3.15) shows that for the supercritical region

$$p_1(n,\alpha,\beta) \le p < \frac{n+2}{n-2}$$

the decay of the solution of nonlinear problem coincides with the decay of the solution of corresponding linear problem, see Chapter One for comparison.

*Remark* 1.3.16. Let us mention that in the case of constant potential in (1.3.11) ($\alpha = 0$ and $\beta = 0$). The critical exponent $p_1(n,\alpha,\beta) = 1 + \frac{4(1-\beta)}{2(n+\alpha)(1-\beta)+\beta(2+\alpha)}$ becomes the Fujita critical exponent $1 + \frac{2}{n}$.

For the subcritical region $1 < p < p_1(n,\alpha,\beta)$, Theorem (1.3.13) shows that the decay estimates is faster and given by the following Corollary.

**Corollary 1.3.17.** *Let the potential $a(t,x) = \lambda(x)\eta(t)$ where $\lambda(x)$, $\eta(t)$ are defined in (1.3.15), (1.3.16) such that condition (1.3.17) satisfies. Assume that*

$$1 < p < p_1(n,\alpha,\beta).$$

*Then the solution of (1.3.11)–(1.3.12) satisfies the following estimates:*

$$\begin{aligned}
\int_{\mathbf{R}^n} e^{c_0(\frac{n+\alpha}{2+\alpha}-2\delta)\frac{|x|^{2+\alpha}}{(1+t)^{1-\beta}}} u^2\, dx &\leq C(1+t)^{(1-\beta)\left(1-\frac{p+1}{p-1}+\frac{p+1}{p-1}\frac{\alpha}{2+\alpha}+\frac{n}{2+\alpha}\right)+\delta}, \\
\int_{\mathbf{R}^n} e^{c_0(\frac{n+\alpha}{2+\alpha}-2\delta)\frac{|x|^{2+\alpha}}{(1+t)^{1-\beta}}} (u_t^2+|\nabla u|^2)\, dx &\leq C(1+t)^{(\beta-1)\left(\frac{p+1}{p-1}-\frac{p+1}{p-1}\frac{\alpha}{2+\alpha}-\frac{n}{2+\alpha}\right)+\delta}, \\
\int_{\mathbf{R}^n} e^{c_0(\frac{n+\alpha}{2+\alpha}-2\delta)\frac{|x|^{2+\alpha}}{(1+t)^{1-\beta}}} |u|^{p+1}\, dx &\leq C(1+t)^{(\beta-1)\left(\frac{p+1}{p-1}-\frac{p+1}{p-1}\frac{\alpha}{2+\alpha}-\frac{n}{2+\alpha}\right)+\delta},
\end{aligned}$$

*where $\delta > 0$ is an arbitrary small number and $t \geq 1$, $c_0$ depends on $\phi_0$ and $\beta$. The constant $C$ depends on $\delta$ and on the initial data $u_0$ and $u_1$.*

The last goal of this thesis is to **study the critical exponent problem** for semi-linear wave equations with space-time dependent potentials. We consider the following Cauchy problem:

$$u_{tt} - \Delta u + a(t,x)u_t = |u|^p, \quad x \in \mathbf{R}^n, \quad t > 0, \tag{1.3.18}$$

$$u(0,x) = \varepsilon u_0(x), \quad u_t(0,x) = \varepsilon u_1(x), \tag{1.3.19}$$

where $\varepsilon > 0$ is a small parameter, $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$ and $1 < p < \infty$ for $n = 1, 2$. The initial data $(u_0, u_1)$ are compactly supported functions and belong to the energy space

$$u_0 \in H^1(\mathbf{R}^n), \quad u_1 \in L^2(\mathbf{R}^n),$$

while the potential $a(t,x)$ is a positive function from $C^1(\mathbf{R}_+, \mathbf{R^n})$. Our objective is to find the critical exponent $p_c(n)$, which is a threshold with the following properties:

- If $1 < p \leq p_c(n)$ all solutions of (1.3.18)–(1.3.19) with positive data in average blow–up in finite time, regardless of the smallness and smoothness of the initial data.
- If $p_c(n) < p < \frac{n+2}{n-2}$ for $n \geq 3$, and $p_c(n) < p$ for $n = 1, 2$ all small data solutions of (1.3.18)-(1.3.19) are global.

It is interesting to compare the dissipative case of (1.3.18)–(1.3.19) with the well known conservative case $a(t,x)=0$. The critical exponent $p_w(n)$ depends only on the space dimension and is given by

$$p_w(n) = \frac{n+1+\sqrt{(n+1)^2+8(n-1)}}{2(n-1)}, \quad n \geq 2.$$

No critical exponent exists for $n=1$, so we set $p_w(1)=\infty$; see Sideris [37]. In fact the number $p_w(n)$, $n \geq 2$, is the positive root of a quadratic equation:

$$(n-1)p^2-(n+1)p-2=0.$$

The proof of these facts, comprising the famous Strauss' conjecture [39], took more than 20 years of work starting with John [18] who verified them for $n=3$. Later Glassey [5] established the conjecture for $n=2$, while Sideris [38] showed the blow-up part for $n \geq 4$. Important contributions were Strauss [40], Zhou [54] and Lindblad and Sogge [23]. The existence part for $n \geq 4$ was finally settled by Georgiev, Lindblad and Sogge [4] and Tataru [41]. Concerning the critical case, Schaeffer [35] showed that $p_w(n)$ belongs to the blow-up region for $n=2,3$. We will call $p_w(n)$ Strauss' critical exponent. Another well-understood case of (1.3.18)–(1.3.19) is $a(t,x)=1$, studied by Todorova and Yordanov [42] and [43]. They found the critical exponent $p_c(n)=1+\frac{2}{n}$, which is the same as Fujita's critical exponent [3] for the heat equation

$$v_t - \Delta v = v^p.$$

Since $p_c(n) < p_w(n)$, we conclude that diffusion effects are sufficiently powerful to shift Strauss' critical exponent to the left. Similar to the parabolic equation, Zhang [53] showed that the critical case $p=p_c(n)$ belongs to the blow-up region. Hence, the global existence and the blow–up results can be summarized as follows: if $p > p_c(n)$, then for sufficiently small data the solutions exist globally, and the energy of all global solutions decays polynomially like $t^{-\frac{n}{4}-\frac{1}{2}}$ as $t \to \infty$; while if $1 < p \leq p_c(n)$, then every solutions with positive data in average blow–up in finite time. Later the global existence of [42], [43] was extended by Ikehata and Tanizawa [17] to certain non-compactly supported initial data. The next step in generalizing problem (1.3.18)-(1.3.19) is the case $a(t, x) = a(x)$, studied by Ikehata, Todorova and Yordanov [12]. These authors have found the critical exponent $p_c(n, \alpha) = 1 + 2/(n - \alpha)$, if $a(x)$ is a radially symmetric function satisfying

$$a \in C^1(\mathbf{R}^n), \quad a(x) > 0 \;\; \forall x \in \mathbf{R}^n,$$

$$a(x) \sim a_0(1 + |x|)^{-\alpha}, \quad |x| \to \infty,$$

with some $\alpha \in [0, 1)$. The global existence for $p > p_c(n, \alpha)$ was derived from sharp estimates of the energy, $L^2$ and $L^{p+1}$ norms. It is relatively straight forward to show blow-up for $1 < p \leq p_c(n, \alpha)$ using the method of Zhang [53]. Let us mention that the slow decay of $a(x)$ is crucial for energy decay. In fact, Mochizuki [27] has shown that scattering theory applies to $u_{tt} - \Delta u + a(x)u_t = 0$ if $\alpha > 1$, so the energy of non-trivial solutions approaches a non-zero constant as $t \to \infty$. Little is known about the case $\alpha = 1$, but it is expected that the decay rate of energy will be an increasing function of $a_0$. The transition between asymptotically parabolic and pure hyperbolic regimes represents a challenging problem which is not well-understood yet. Hence the critical exponent will be very difficult to find when $\alpha = 1$.

In this thesis **we derive an upper bound on the critical exponent for problem (1.3.18)–(1.3.19)** with radially symmetric potentials $a(t, x)$, such that

$$a \in C^1(\mathbf{R}_+ \times \mathbf{R}^n), \quad a(t, x) > 0 \;\; \forall (t, x) \in \mathbf{R}_+ \times \mathbf{R}^n, \tag{1.3.20}$$

$$a(t, x) \sim a_0(1 + |x|)^{-\alpha}(1 + t)^{-\beta}, \quad t, |x| \to \infty, \tag{1.3.21}$$

with $a_0 > 0$, $\alpha \in [0, 1)$, $\beta \in (-1, 1)$ and $0 < \alpha + \beta < 1$. The threshold is given explicitly by

$$p_c(n, \alpha, \beta) = 1 + \frac{4(\beta + 1)}{2(n - \alpha)(\beta + 1) - \beta(2 - \alpha)}. \tag{1.3.22}$$

Notice that $p_c(n,\alpha,\beta) > p_c(n,\alpha)$ for $\beta > 0$.

We used the new techinique, which is a strengthen of the multiplier method that was developed by Todorova and Yordanov [44] for the linear wave equations with space dependent potential. The existence of the foucusing nonlinearity produced weighted space–time norm and weighted space norm. It was delicate to estimate the weighted space norm, with some weight on the left so we will be able to gain some decay on the left hand side, so we are able to control the space–time dependent norm.

**Theorem 1.3.18.** *Let $p_c(n,\alpha,\beta)$ be defined in (1.3.22) and $a(t,x)$ satisfy conditions (1.3.20) and (1.3.21) with $\alpha \in [0,1)$, $\beta \in (-1,1)$, and $0 < \alpha+\beta < 1$. If $p$ satisfies $p_c(n,\alpha,\beta) < p < \frac{n+2}{n-2}$ for $n \geq 3$ and $p_c(n,\alpha,\beta) < p < \infty$ for $n = 1,2$, there exists a number $\varepsilon_0 > 0$, such that problem (1.3.18)–(1.3.19) has a unique solution $u \in X_1(0,\infty)$ for any $0 < \varepsilon < \varepsilon_0$. Moreover, the following estimates hold:*

$$\begin{aligned}
\int_{\mathbf{R^n}} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} u^2\, dx &\leq Ct^{\beta+(\beta+1)(2\delta+\frac{\alpha}{2-\alpha}-m(\lambda))}, \\
\int_{\mathbf{R^n}} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} (u_t^2+|\nabla u|^2)\, dx &\leq Ct^{(\beta+1)(2\delta-m(\lambda))-1}, \\
\int_{\mathbf{R^n}} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} |u|^{p+1}\, dx &\leq Ct^{-(\xi+1)+p(\beta+1)(2\delta-m(\lambda))},
\end{aligned}$$

*for all $t \geq 1$. Here $\delta > 0$ can be arbitrarily small, while $\xi > 0$ is given by*

$$\xi = (\beta+1)\left(\frac{(p-1)(n-\alpha)-2}{2-\alpha} - \delta(p-1)\right) - \beta\left(\frac{p-1}{2}\right).$$

*Two-sided inequalities for $\phi \in C^2(\mathbf{R}^n)$ are shown in Proposition (2.1.1); in particular, $\phi(x) \geq \phi_0|x|^{2-\alpha}$ with $\phi_0 > 0$.*

**Corollary 1.3.19.** *Under the assumptions of Theorem (1.3.18),*

$$\begin{aligned}
\int_{\mathbf{R^n}} u^2\ dx &\leq Ct^{\beta+(\beta+1)(2\delta+\frac{2\alpha-n}{2-\alpha})},\\
\int_{\mathbf{R^n}} (u_t^2+|\nabla u|^2)dx &\leq Ct^{(\beta+1)(2\delta-\frac{n-\alpha}{2-\alpha})-1},\\
\int_{\mathbf{R^n}} |u|^{p+1}\ dx &\leq Ct^{-(\xi+1)+p(\beta+1)(2\delta-\frac{n-\alpha}{2-\alpha})},
\end{aligned}$$

*for large* $t >> 1$.

**Corollary 1.3.20.** *Let the assumptions of Theorem (1.3.18) hold and fix* $\epsilon > 0$. *For every* $\delta > 0$, *the solution of (1.3.18)–(1.3.19) satisfies*

$$\int_{\phi(x)\geq t^{(1+\beta)(1+\epsilon)}} (u_t^2+|\nabla u|^2)dx \leq Ce^{-(m(\lambda)-2\delta)t^{\epsilon(1+\beta)}},$$

*where* $t >> 1$.

Thus, the local energy in $\{x : \phi(x) \geq t^{(1+\epsilon)(1+\beta)}\}$ decays exponentially fast as $t \to \infty$. This observation confirms that small-data solutions of (1.3.18)–(1.3.19) have parabolic asymptotic profiles.

An interesting observation is that the decay rate of $\|u\|_{L^{p+1}}^{p+1}$ is larger than the decay that can be derived by the standard interpolation inequality, namely the Gagliargo–Nirenberg inequality and the decay estimates of $\|u\|_{L^2}^2$ and $\|\nabla u\|_{L^2}^2$.

# Chapter 2

# Linear Dissipative Wave Equations with Potential $a(t, x)$

Consider the Cauchy problem for the linear dissipative wave equation,

$$u_{tt} - \Delta u + a(t,x)u_t = 0, \quad x \in \mathbf{R}^n, \quad t > 0, \tag{2.0.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{2.0.2}$$

Concerning the initial data we assume that they are compactly supported and belong to the energy space

$$u_0 \in H^1(\mathbf{R}^n), \quad u_1 \in L^2(\mathbf{R}^n), \quad u_0(x) \text{ and } u_1(x) = 0 \text{ for } |x| > R.$$

The potential $a(t,x) = \lambda(x)\eta(t)$ is a $C^1(\mathbf{R}_+, \mathbf{R^n})$ function, which is radially symmetric with respect to $x$, where $\lambda(x)$ and $\eta(t)$ satisfy the following conditions:

$$\lambda_0(1+|x|)^\alpha \leq \quad \lambda(x) \quad \leq \lambda_1(1+|x|)^\alpha, \quad \alpha \in (0,\infty), \tag{2.0.3}$$

$$\eta_0(1+t)^\beta \leq \quad \eta(t) \quad \leq \eta_1(1+t)^\beta, \quad \beta \in (-1,1). \tag{2.0.4}$$

for all $(t, x) \in (\mathbf{R}_+, \mathbf{R}^n)$, $\lambda_0$, $\lambda_1$, $\eta_0$, $\eta_1 > 0$, such that

$$\frac{\eta_0^2}{\eta_1^2(1-\beta)^2} > \frac{\lambda_1}{3\lambda_0(n+\alpha)(2+\alpha)}. \tag{2.0.5}$$

We impose additional conditions on $\eta(t)$:

$$\begin{aligned} \left|\frac{d^k\eta(t)}{dt^k}\right| &\leq C_k \quad \frac{\eta(t)}{(1+t)^k}, \qquad k \in \mathbf{N}, \\ \frac{d\eta(t)}{dt} &\leq \quad 0. \end{aligned}$$

For all $t$ sufficiently large. It is a well known result that (2.0.1) admits a unique weak solution with regularity

$$u \in C((0,\infty), H^1(\mathbf{R}^n)), \quad u_t \in C((0,\infty), L^2(\mathbf{R}^n)),$$

and compact support

$$u(t, x) = 0 \text{ for } |x| > t + R.$$

**This chapter is organized as follows.** In section (2.1) the modified equation for $v = uw^1$ and the main weighted energy identities are computed and estimated. In section (2.2) the sufficient conditions on the weights are stated. In section (2.3) we present the good weights and prove that they satisfy the conditions of the previous section. In section (2.4) the Theorems and the Corollaries are proved.

As we mentioned in the introduction, in this work we are going to use a generalization of the new technique presented by Todorova and Yordanov [44].

## 2.1 Deriving the Weighted Energy Identities

Before deriving the weighted energy we have to present some important information. Due to the diffusion phenomenon, the approximate solution $w$ of (2.0.1) with space-time dependent potential $a(t,x) = \lambda(x)\eta(t)$, is the solution of the corresponding parabolic equation,

$$a(t,x)w_t - \Delta w = 0, \quad x \in \mathbf{R}^n, \quad t > 0, \tag{2.1.1}$$

This equation is simpler when the potential $a(t,x)$ is only a space dependent, to be able to produce this we define

$$w_T = \eta(t)w_t, \text{ but } \ w_t = w_T T'$$

Hence we produced a new time parameter $T(t)$ such that

$$T(t) = \int_0^t \frac{ds}{\eta(s)}, \tag{2.1.2}$$

where $T \to \infty$ as $t \to \infty$; in fact, $T \sim (1+t)^{1-\beta}$, $T' = \frac{1}{\eta(t)} \sim (1+t)^{-\beta}$ as $t \to \infty$. The corresponding equation in $T$ can be written as

$$\lambda(x)w_T - \Delta w = 0, \quad x \in \mathbf{R}^n, \quad t > 0. \tag{2.1.3}$$

An approximate solution is

$$w(T,x) = T^{-m}e^{-\gamma\frac{\phi(x)}{T}}, \tag{2.1.4}$$

with suitably chosen constants $m$, $\gamma$, where $\phi(x)$ is a solution of the Poisson equation

$$\Delta\phi(x) = \lambda(x), \quad x \in \mathbf{R}^n. \tag{2.1.5}$$

Here $\lambda(x)$ is given by (2.0.3) and $\phi(x)$ has the following properties:

$$\begin{aligned}
&(a1) \qquad \phi(x) \geq 0 \ \text{ for all } \ x \in \mathbf{R}^n, \\
&(a2) \qquad \phi(x) = O(|x|^{2+\alpha}) \ \text{ for large } \ |x|, \\
&(a3) \qquad m(\lambda) = \liminf_{x\to\infty} \frac{\lambda(x)\phi(x)}{|\nabla\phi(x)|^2} \ > \ 0.
\end{aligned}$$

Such solutions $\phi(x)$ exist in many cases, including radial coefficients $\lambda(x)$ which behave like $|x|^\alpha$ as $|x| \to \infty$, see Appendix $C$.

**Proposition 2.1.1.** *Let $\lambda(x)$ is radially symmetric function in $C^1(\mathbf{R}^n)$, $n \geq 3$, which satisfies (2.0.3). Then equation (2.1.5) admits a solution $\phi \in C^2(\mathbf{R}^n)$, such that*

$$\begin{aligned}
&(A1) \qquad \phi_0(1+|x|)^{2+\alpha} \leq \phi(x) \leq \phi_1(1+|x|)^{2+\alpha}, \\
&(A2) \qquad m(\lambda) > 0,
\end{aligned}$$

*where $\phi_0$ and $\phi_1$ are positive constants. In the special case*

$$\lambda(x) \sim \lambda_2|x|^\alpha, \quad |x| \to \infty, \tag{2.1.6}$$

*with $\lambda_2 > 0$, equation (2.1.5) has a solution with the following properties:*

$$\begin{aligned}
&(A3) \qquad \phi(x) \sim \frac{\lambda_2}{(2+\alpha)(n+\alpha)}|x|^{2+\alpha}, \quad |x| \to \infty, \\
&(A4) \qquad m(\lambda) = \frac{n+\alpha}{2+\alpha}.
\end{aligned}$$

Since $v = w^{-1}u$ is more stable than $u$ itself, therefore we need to derive the modified equation for $v$, and then derive the weighted energy identities, by multiplying the modified equation with the multipliers $Pv_t + wv$ where the weights $P$, $w$ have to be defined in a way to insure that the weighted energy is non-increasing and positive definite.

The following Lemma is already presented in Todorova and Yordanov [44], but since we are using it, for our convenience we are going to present it here again.

**Lemma 2.1.2.** *Consider a general damped wave equation*

$$u_{tt} - \triangle u + a(t,x)u_t + b \cdot \nabla u + cu = 0, \tag{2.1.7}$$

*where the coefficients* $a(t,x)$ *is a* $C^1(\mathbf{R}_+, \mathbf{R}^n)$, $b = (b_1, b_2, \ldots, b_n)$ *and* $c$ *are* $C^1$ *functions of* $t$, $x$ *on* $\mathbf{R}_+ \times \mathbf{R}^n$. *If* $w$ *is* $C^2(\mathbf{R}_+, \mathbf{R}^n)$, *this form is invariant under the following transformation* $v = w^{-1}u$ *where* $w$ *is the approximate solution of (2.1.3), we have the following transformed wave equation:*

$$v_{tt} - \triangle v + \hat{a}v_t + \hat{b} \cdot \nabla v + \hat{c}v = 0, \tag{2.1.8}$$

*where*

$$\begin{aligned} \hat{a} &= a(t,x) + 2w_t w^{-1}, \\ \hat{b} &= b - 2w^{-1} \cdot \nabla w, \\ \hat{c} &= w^{-1}\Big(w_{tt} - \triangle w + a(t,x)w_t + b \cdot \nabla w + cw\Big). \end{aligned} \tag{2.1.9}$$

*Proof.* See Appendix $B$. ☐

Now we need to derive the weighted identities as addressed before. We multiply the transformed equation (2.1.8) by $Pv_t + wv$ and integrate where $P, w \in C^2((0,\infty) \times \mathbf{R}^n)$ are the positive weights that we have to define later.

**Proposition 2.1.3.** *If* $u$ *is the solution (2.1.7) with compactly supported data*

$$u_0 \in H^2(\mathbf{R}^n), \qquad u_1 \in H^1(\mathbf{R}^n).$$

*Assume that $P$ and $w > 0$ are $C^2$ – functions. Then*

$$\frac{d}{dt}E(v_t, \nabla v, v) + F(v_t, \nabla v) + G(v) = 0, \tag{2.1.10}$$

*where*

$$\begin{aligned}
E(v_t, \nabla v, v) &= \frac{1}{2}\int_{\mathbf{R}^n} \Big( P(v_t^2 + |\nabla v|^2) + 2wv_tv + (\hat{c}P - w_t + \hat{a}w)v^2 \Big)\ dx, \\
F(v_t, \nabla v) &= \frac{1}{2}\int_{\mathbf{R}^n} (-P_t + 2\hat{a}P - 2w)v_t^2\ dx \\
&+ \int_{\mathbf{R}^n} (\nabla P + \hat{b}P) \cdot v_t \nabla v\ dx \\
&+ \frac{1}{2}\int_{\mathbf{R}^n} (-P_t + 2w)|\nabla v|^2\ dx, \\
G(v_t, \nabla v) &= \frac{1}{2}\int_{\mathbf{R}^n} [w_{tt} - \Delta w - (\hat{a}w)_t - \nabla \cdot (\hat{b}w) + 2\hat{c}w - (\hat{c}P)_t]v^2\ dx.
\end{aligned}$$

*The coefficients $\hat{a}, \hat{b}$ and $\hat{c}$ are given by (2.1.9).*

*Proof.* See Appendix $B$ with $k = 0$. □

The damped wave equation that we have has

$$a(t,x) = \lambda(x)\eta(t), \qquad b = 0, \qquad \text{and} \qquad c = 0,$$

so the transformed coefficients will be:

$$\begin{aligned}
\hat{a} &= \lambda(x)\eta(t) + 2w_t w^{-1}, \\
\hat{b} &= -2w^{-1} \cdot \nabla w, \\
\hat{c} &= w^{-1}(w_{tt} - \triangle w + \eta(t)\lambda(x)w_t).
\end{aligned} \tag{2.1.11}$$

Now we rewrite the identities in Proposition (2.1.3) in terms of the new coefficients as in (2.1.11). For simplicity we are going to leave $\hat{c}$ and only substitute $\hat{a}$, $\hat{b}$. After some simple calculations we will have the following identities:

$$\begin{aligned} E(v_t, \nabla v, v) &= \frac{1}{2}\int_{\mathbf{R}^n} \left(P(v_t^2 + |\nabla v|^2) + 2wv_tv + (\hat{c}P + w_t + aw)v^2\right)\, dx, \\ F(v_t, \nabla v) &= \frac{1}{2}\int_{\mathbf{R}^n} (-P_t + 2aP + 4P(\ln w)_t - 2w)v_t^2\, dx \\ &+ \int_{\mathbf{R}^n} (\nabla P - 2P\nabla \ln w)\cdot v_t \nabla v\, dx \\ &+ \frac{1}{2}\int_{\mathbf{R}^n} (-P_t + 2w)|\nabla v|^2\, dx, \\ G(v) &= \int_{\mathbf{R}^n} (\hat{c}w - (\hat{c}P)_t)v^2\, dx, \end{aligned}$$

such that the weighted energy identity is satisfied:

$$\frac{d}{dt}E(v_t, \nabla v, v) + F(v_t, \nabla v) + G(v) = 0,$$

for initial data $u_0 \in H^2(\mathbf{R^n})$ and $u_1 \in H^1(\mathbf{R^n})$.

## 2.2 Sufficient Conditions on the Weights

What are the conditions on the damping and the weights to insure $F(v_t, \nabla v) \geq 0$ and $G(v_t, \nabla v) \geq 0$, so that the weighted energy is non–increasing and positive definite?

**Proposition 2.2.1.** *Let $P$ and $w$ be positive weights and $\hat{c}$ be defined at (2.1.11). Assume that*

$(i)$ $\quad \hat{c} \geq 0, \quad \hat{c}_t \leq 0,$

$(ii)$ $\quad -P_t + w \geq 0,$

$(iii)$ $\quad (-P_t + 2w)(-P_t + 2aP + 4P(\ln w)_t - 2w) \geq (\nabla P - 2P\nabla \ln w)^2,$

*for all $t \geq t_0$ sufficiently large, $|x| \leq t + R$. If $u$ is a solution of (2.0.1), $E(v_t, \nabla v, v)$ is a non-increasing function of time:*

$$\frac{1}{2}\int_{\mathbf{R}^n}\left(P(v_t^2 + |\nabla v|^2) + 2wv_t v + (\hat{c}P + w_t + aw)v^2\right)\,dx \leq E(v_t, \nabla v, v)|_{t=t_0}$$

*for all $t \geq t_0$.*

*Proof.* If the data are compactly supported and satisfy $u_0 \in H^2$ and $u_1 \in H^1$, identity (2.1.10) holds. Notice that conditions $(i)$ and $(ii)$ imply

$$\hat{c}w - (\hat{c}P)_t = \hat{c}(w - P_t) - \hat{c}_t P \geq 0.$$

Hence $G(v) \geq 0$. Condition $(iii)$ and $-P_t + 2w \geq 0$, which follows from $(ii)$, guarantee that the quadratic form $F(v_t, \nabla v) \geq 0$. Thus (2.1.10) yields $\frac{d}{dt}E(v_t, \nabla v, v) \leq 0$ or $E(v_t, \nabla v, v) \leq E_0 = E(v_t, \nabla v, v)|_{t=t_0}$.

For any compactly supported data $u_0 \in H^1$ and $u_1 \in L^2$, there exist compactly supported $C^\infty$-sequences $u_0^{(k)} \to u_0$ in $H^1$ and $u_1^{(k)} \to u_1$ in $L^2$. Denote the corresponding solutions of (2.0.1) by $u^{(k)}$ and their weighted energy by $E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)})$. The first part of the proof shows that

$$E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)}) \leq E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)})|_{t=t_0}. \tag{2.2.1}$$

Since the weights $P$ and $w$ are continuous functions, the weighted energy is bounded by the standard energy:

$$\begin{aligned} & |E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)}) - E(v_t, \nabla v, v)| \\ \leq \;& c(T^*)\left(\|u_t^{(k)} - u_t\|_{L^2}^2 + \|\nabla u^{(k)} - \nabla u\|_{L^2}^2\right), \end{aligned}$$

whenever $t \in [0, T^*]$ using the Poincaré inequality. It is easy to be proved that the weak solutions of problem (2.0.1) satisfy the energy estimate

$$\|u_t^{(k)} - u_t\|_{L^2}^2 + \|\nabla u^{(k)} - \nabla u\|_{L^2}^2 \ \leq \ c\left(\|u_1^{(k)} - u_1\|_{L^2}^2 + \|\nabla u_0^{(k)} - \nabla u_0\|_{L^2}^2\right).$$

Hence

$$\begin{aligned} & |E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)}) - E(v_t, \nabla v, v)| \\ \leq \ & c(T^*)\left(\|u_1^{(k)} - u_1\|_{L^2}^2 + \|\nabla u_0^{(k)} - \nabla u_0\|_{L^2}^2\right). \end{aligned}$$

We use the latter estimate to pass to the limit as $k \to \infty$ in (2.2.1):

$$E(v_t, \nabla v, v) = \lim_{k\to\infty} E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)}) \leq \lim_{k\to\infty} E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)})|_{t=t_0} = E_0.$$

This completes the proof for general data. $\square$

**Proposition 2.2.2.** *Let $P$ and $w$ be positive weights satisfying conditions (i)–(iii) in Proposition (2.2.1), and the following condition,*

$$(iv) \qquad (1-\epsilon)P\left(w_t + (1-\epsilon)aw\right) > w^2,$$

*where $\epsilon \in (0,1)$. Assume that $a(t,x) = \lambda(x)\eta(t)$, where $\lambda(x)$ and $\eta(t)$ are defined in (2.0.3) and (2.0.4) respectively, Assume also that condition (2.0.5) satisfied. Then*

$$\begin{aligned} \int_{\mathbf{R}^n} P(v_t^2 + |\nabla v|^2)\, dx \ &\leq \ CE_0, \\ \int_{\mathbf{R}^n} awv^2\, dx \ &\leq \ CE_0, \end{aligned}$$

*for $t \geq t_0$.*

*Proof.* We begin with the result of Proposition (2.2.1)

$$\int_{\mathbf{R}^n} \Big( P(v_t^2 + |\nabla v|^2) + 2wv_t v + (\hat{c}P + w_t + aw)v^2 \Big) \; dx \leq 2E_0.$$

This equation can be rewritten as

$$\begin{aligned} &\int_{\mathbf{R}^n} \Big( (1-\epsilon)P(v_t^2 + |\nabla v|^2) + 2wv_t v + (\hat{c}P + w_t + (1-\epsilon)aw)v^2 \Big) dx \\ &+ \int_{\mathbf{R}^n} \epsilon P(v_t^2 + |\nabla v|^2) dx + \int_{\mathbf{R}^n} \epsilon awv^2 dx \leq 2E_0, \end{aligned}$$

for any $\epsilon \in (0,1)$. Condition $(iv)$ insures the following quadratic form

$$(1-\epsilon)Pv_t^2 + 2wv_t v + \big(w_t + (1-\epsilon)aw\big)v^2 \geq 0,$$

is positive definite □

**Proposition 2.2.3.** *Let $P$ and $w$ be positive weights satisfying conditions (i)–(iv). If*

$$(v) \qquad Pw^{-3}(w_t^2 + |\nabla w|^2) \leq Ca(t,x),$$

*holds for some $C > 0$, $a(t,x) = \lambda(x)\eta(t)$, where $\lambda(x)$ and $\eta(t)$ are defined by (2.0.3) and (2.0.4) respectively, such that condition (2.0.5) satisfied. Then*

$$\begin{aligned} \int_{\mathbf{R}^n} aw^{-1}u^2 \; dx &\leq CE_0, \\ \int_{\mathbf{R}^n} Pw^{-2}(u_t^2 + |\nabla u|^2) \; dx &\leq CE_0, \end{aligned} \tag{2.2.2}$$

*for all $t \geq t_0$.*

*Proof.* Substitute $v = w^{-1}u$ in the second estimate of Proposition (2.2.2) so we have

$$\int_{\mathbf{R}^n} aww^{-2}u^2 dx \leq CE_0,$$
$$\int_{\mathbf{R}^n} aw^{-1}u^2 dx \leq CE_0.$$

To show the second estimate, since $v = w^{-1}u$

$$v_t^2 = (-w^{-2}w_t u + w^{-1}u_t)^2 \geq \frac{1}{2}w^{-2}u_t^2 - 3w^{-4}w_t^2 u^2,$$

hence

$$u_t^2 \leq 2w^2 v_t^2 + 6w^{-2}w_t^2 u^2$$

and

$$|\nabla v|^2 = (-w^{-2}\nabla w u + w^{-1}\nabla u)^2 \geq \frac{1}{2}w^{-2}|\nabla u|^2 - 3w^{-4}|\nabla w|^2 u^2,$$

hence

$$|\nabla u|^2 \leq 2w^2|\nabla v|^2 + 6w^{-2}|\nabla w|^2 u^2.$$

Thus

$$\begin{aligned}
\frac{1}{2}Pw^{-2}(u_t^2 + |\nabla u|^2) &\leq \frac{1}{2}Pw^{-2}(2w^2 v_t^2 + 6w^{-2}w_t^2 u^2 + 2w^2|\nabla v|^2 + 6w^{-2}|\nabla w|^2 u^2), \\
&\leq P(v_t^2 + |\nabla v|^2) + 3Pw^{-4}(w_t^2 + |\nabla w|^2)u^2.
\end{aligned}$$

Integrating this inequality and applying the estimate in Proposition (2.2.3), we obtain

$$\frac{1}{2}\int_{\mathbf{R}^n} Pw^{-2}(u_t^2 + |\nabla u|^2)\ dx \leq CE_0 + 3\int_{\mathbf{R}^n} Pw^{-4}(w_t^2 + |\nabla w|^2)u^2\ dx.$$

Using condition $(v)$

$$\frac{1}{2}\int_{\mathbf{R}^n} Pw^{-2}(u_t^2+|\nabla u|^2)\ dx \le C_1E_0 + C_2\int_{\mathbf{R}^n} a(t,x)w^{-1}u^2\ dx.$$

Using estimate (2.2.2), the proof is completed. □

## 2.3 Finding Good Weights

Our main aim in this section is to find the family of weights $P$ and $w$ satisfying conditions $(i)-(v)$ listed in the previous section. As discussed before, $w$ will be an approximate solution of equation (2.1.3)

We are interested in the Poisson equation

$$\Delta\phi(x) = \lambda(x), \quad x \in \mathbf{R}^n.$$

We assume that there exists a solution $\phi(x)$ with properties $(a1)-(a3)$ as given in Section (2.1). Given a small $\delta \in (0, \frac{1}{2}m(\lambda))$ and a large $S_0 > 0$, set

$$m = m(\lambda) - 2\delta, \quad S(x) = (m(\lambda)-\delta)\phi(x) + S_0. \tag{2.3.1}$$

The two weights are defined as

$$w(T,x) = T^{-m}e^{-\frac{S(x)}{T}}, \quad P(T,x) = \frac{3}{4}\left(\frac{6}{T}+\frac{S(x)}{T^2}\right)^{-1}\frac{w(T,x)}{T'}. \tag{2.3.2}$$

Note that $S$, $P$, $w > 0$ on $(\mathbf{R}_+ \times \mathbf{R}^n)$. $T$ is defined above (2.1.2)

**Lemma 2.3.1.** *Define $m$ and $S$ by (2.3.1) and assume that $m(\lambda)$ and $\phi(x)$ have properties $(a1)-(a3)$. There exists $S_0 > 0$ such that*

$$
\begin{aligned}
&(S1) \qquad \Delta S(x) = (m+\delta)\lambda(x) \ \ \textit{for all} \ \ x \in \mathbf{R}^n, \\
&(S2) \qquad S(x) = O(|x|^{2+\alpha}) \ \ \textit{for large} \ \ |x|, \\
&(S3) \qquad \left(1 - \frac{\delta}{2m(\lambda)}\right)\lambda(x)S(x) - |\nabla S(x)|^2 \geq 0 \ \ \textit{for all} \ \ x \in \mathbf{R}^n.
\end{aligned}
$$

*Proof.* See Todorova, Yordanov [44] □

Now we have to make sure that the weights (2.3.2) satisfy conditions $(i)-(v)$

**Proposition 2.3.2.** *Assume that $\phi(x)$ satisfies $(a1)-(a3)$. Let $P$ and $w$ be defined in (2.3.2) with $m$ and $S$ defined at (2.3.1). Then conditions $(i)-(v)$ hold for sufficiently large $t \geq t_0$.*

*Proof.* $(i)$ Using Lemma (A.3), $\lambda_0 \leq \lambda_0(1+|x|)^\alpha \leq \lambda(x)$ and $T \sim (1+t)^{1-\beta}$ we have

$$
\begin{aligned}
\hat{c} \ &\geq \ \frac{-mT'' + \delta\lambda_0}{T} + \frac{S(x)T'' + k\delta\lambda_0 S(x) - \frac{2S(x){T'}^2}{T}}{T^2} \\
&\geq \ \frac{-m(1+t)^{-1-\beta} + \delta\lambda_0}{(1+t)^{1-\beta}} + \frac{S(x)(1+t)^{-1-\beta} + k\delta\lambda_0 S(x) - \frac{2S(x)(1+t)^{-2\beta}}{(1+t)^{1-\beta}}}{(1+t)^{2(1-\beta)}} \\
&\geq \ -\frac{m}{(1+t)^2} + \frac{\delta\lambda_0}{(1+t)^{1-\beta}} + S(x)\left((1+t)^{-3+\beta} + \frac{k\delta\lambda_0}{(1+t)^{2(1-\beta)}} - 2(1+t)^{-3+\beta}\right) \\
&\geq \ -\frac{m}{(1+t)^2} + \frac{\delta\lambda_0}{(1+t)^{1-\beta}} + S(x)\left(\frac{-1}{(1+t)^{3-\beta}} + \frac{k\delta\lambda_0}{(1+t)^{2(1-\beta)}}\right),
\end{aligned}
$$

so we conclude that $\hat{c} \geq 0$ for sufficiently large $t$, whenever $\beta > -1$. It is similar to show that $\hat{c}_t \leq 0$.

$(ii)$ Using Lemma (A.4) and $T \sim (1+t)^{1-\beta}$ we have

$$\begin{aligned}
-P_t + w &\geq \left( \frac{m+7}{(1+t)^{1-\beta}} + \frac{(1+t)^{-1-\beta}}{(1+t)^{-2\beta}} \right) (1+t)^{-\beta} P \\
&= \left( \frac{m+7}{1+t} + (1+t)^{-1} \right) P \\
&= \left( \frac{m+8}{1+t} \right) P \\
&\geq 0.
\end{aligned}$$

$(iii)$ We can estimate the first factor on the left side by condition $(ii)$:

$$\begin{aligned}
-P_t + 2w &= (-P_t + w) + w \\
&\geq w.
\end{aligned}$$

Using Lemma (A.4)

$$\begin{aligned}
-P_t + 2aP + 4P(\ln w)_t - 2w &\geq \left( \frac{-3m-17}{(1+t)^{1-\beta}} + \frac{(1+t)^{-1-\beta}}{(1+t)^{-2\beta}} + \frac{2\lambda(x)\eta(t)}{(1+t)^{-\beta}} \right) \frac{P}{(1+t)^{\beta}} \\
&= \left( -\frac{3m+17}{1+t} + \frac{1}{1+t} + 2\lambda(x)\eta(t) \right) P \\
&= \left( -\frac{3m+16}{1+t} + 2\lambda(x)\eta(t) \right) P \\
&\geq a(t,x)P.
\end{aligned}$$

For sufficiently large $t$. Hence the left hand side

$$
\begin{aligned}
\Big(-P_t+2aP+4P(\ln w)_t-2w\Big)\Big(-P_t+2w\Big) &\geq Pw\lambda(x)\eta(t)\\
&= P^2\frac{w}{P}\lambda(x)\eta(t)\\
&= P^2\frac{4}{3}\Big(\frac{6}{T}+\frac{S(x)}{T^2}\Big)T'\lambda(x)\eta(t)\\
&\geq P^2\frac{4}{3}\Big(\frac{6T+S(x)}{T^2}\Big)T'\lambda(x)\eta(t)\\
&\geq \frac{4}{3}\frac{\lambda(x)S(x)}{T^2}P^2. \qquad (2.3.3)
\end{aligned}
$$

For sufficiently large $t$, equation (2.3.3) follows since $T'\eta(t)=1$.

It remains to bound the right side of inequality $(iii)$. Using Lemma (A.4) and condition $(S3)$, $|\nabla S(x)|^2 \leq \lambda(x)S(x)$,

$$
\begin{aligned}
(\nabla P-2P\nabla\ln w)^2 &= \frac{(5T+S(x))^2|\nabla S(x)|^2}{T^2(6T+S(x))^2}P^2\\
&\leq \frac{\lambda(x)S(x)(5T+S(x))^2}{T^2(6T+S(x))^2}P^2\\
&\leq \frac{\lambda(x)S(x)}{T^2}P^2,
\end{aligned}
$$

compare this upper bound with the lower bound (2.3.3), hence condition $(iii)$ holds.

(iv) Rewrite condition $(iv)$ as

$$
\begin{aligned}
\frac{w_t}{w}+(1-\epsilon)a(t,x) &\geq \frac{w}{(1-\epsilon)P},\\
(1-\epsilon)a(t,x) &\geq \frac{w}{(1-\epsilon)P}-\frac{w_t}{w}.
\end{aligned}
$$

Using the weights $P$, $w$ as defined in (2.3.2) we have,

$$\begin{aligned}(1-\epsilon)a(t,x) &\geq \frac{4}{3(1-\epsilon)}\left(\frac{6}{T}+\frac{S(x)}{T^2}\right)T' - \left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)T' \\ &= \frac{1+3\epsilon}{3(1-\epsilon)}\frac{S(x)}{T^2}T' + \frac{8+m}{T}T',\end{aligned}$$

choosing $\epsilon \approx 1$, we have to prove the following inequality:

$$a(t,x) \geq \frac{S(x)}{3T^2}T' + \frac{8+m}{T}T',$$

since $\frac{T'}{T} \sim \frac{1}{t}$, for sufficiently large $t$ we have

$$a(t,x) \geq \frac{S(x)}{3T^2}T' + \delta_1 \geq \frac{S(x)}{3T^2}T' + \frac{8+m}{T}T', \text{where } \delta_1 \text{ is arbitrary small positive number.}$$

Since $\delta_1 > 0$ is arbitrary small number we have to prove that

$$\begin{aligned}a(t,x) &> \frac{S(x)}{3T^2}T' \\ \lambda(x)\eta(t) &> \frac{1}{3\eta(t)}\frac{S(x)}{\left(\int_0^t \frac{ds}{\eta(s)}\right)^2}.\end{aligned}$$

We separate the variables we get

$$\eta^2(t)\left(\int_0^t \frac{ds}{\eta(s)}\right)^2 > \frac{1}{3}\frac{S(x)}{\lambda(x)}.$$

Using $\lambda(x)$, $\eta(t)$ as defined by (2.0.3), (2.0.4) respectively, and Corollary (C.7) from Appendix $(C)$ we have

$$\begin{aligned} \eta_0^2(1+t)^{2\beta}\frac{(1+t)^{2-2\beta}}{\eta_1^2(1-\beta)^2} &> \frac{1}{3}\frac{\lambda_1|x|^{2+\alpha}}{\lambda_0|x|^\alpha}\frac{1}{(n+\alpha)(2+\alpha)} \\ \frac{\eta_0^2}{\eta_1^2(1-\beta)^2}(1+t)^2 &> \frac{\lambda_1}{3\lambda_0(n+\alpha)(2+\alpha)}|x|^2. \end{aligned} \tag{2.3.4}$$

By using the finite speed of propagation the above inequality (2.3.4) is reduced to

$$\frac{\eta_0^2}{\eta_1^2(1-\beta)^2} > \frac{\lambda_1}{3\lambda_0(n+\alpha)(2+\alpha)}$$

. $(v)$ Using Lemma (A.4) and $T \sim (1+t)^{1-\beta}$ we have

$$\begin{aligned} Pw^{-3}\left(w_t^2+|\nabla w|^2\right) &\leq C\left(\frac{m^2(1+t)^{-\beta}}{(1+t)^{1-\beta}}+\frac{(1+t)^{2+\alpha}}{(1+t)^{2-2\beta}}(1+t)^{-\beta}+\lambda(x)\eta(t)\right) \\ &\leq C\left(\frac{m^2}{1+t}+(1+t)^{\alpha+\beta}+\lambda(x)\eta(t)\right) \\ &\leq Ca(t,x). \end{aligned}$$

$\square$

## 2.4 Proofs of Main Theorems and Corollaries

*Proof.* (**Proof of Theorem 1.3.1**) We apply Proposition (2.2.3), the definition of $m$ and $S(x)$ as given in (2.3.1), the weights defined by (2.3.2) and $\lambda(x) \geq \lambda_0$:

$$\begin{aligned}
\int_{\mathbf{R}^n} \lambda(x)\eta(t)T^m e^{\frac{S(x)}{T}} u^2\, dx &\leq CE_0, \\
\int_{\mathbf{R}^n} \lambda_0 T^{(m(\lambda)-2\delta)} e^{(m(\lambda)-\delta)\frac{\phi(x)}{T}} u^2\, dx &\leq CE_0 T', \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-\delta)\frac{\phi(x)}{T}} u^2\, dx &\leq CE_0 T' T^{2\delta - m(\lambda)}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} u^2\, dx &\leq CE_0(1+t)^{-\beta}(1+t)^{(1-\beta)(2\delta - m(\lambda))}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} u^2\, dx &\leq CE_0(1+t)^{(1-\beta)(2\delta - m(\lambda))-\beta}.
\end{aligned}$$

To show the second estimate, we have

$$\begin{aligned}
\int_{\mathbf{R}^n} P w^{-2}(u_t^2 + |\nabla u|^2)dx &\leq CE_0, \\
\int_{\mathbf{R}^n} \frac{3}{4}\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1} \frac{T^m}{T'} e^{\frac{S(x)}{T}} (u_t^2 + |\nabla u|^2)dx &\leq CE_0, \\
\int_{\mathbf{R}^n} \left(\frac{1}{T} + \frac{\phi(x)}{T^2}\right)^{-1} e^{(m(\lambda)-\delta)\frac{\phi(x)}{T}} (u_t^2 + |\nabla u|^2)dx &\leq CE_0 T' T^{-m(\lambda)+2\delta}.
\end{aligned}$$

Now

$$\left(\frac{1}{T} + \frac{\phi(x)}{T^2}\right)^{-1} = T\left(1 + \frac{\phi(x)}{T}\right)^{-1} \geq T e^{-\delta\frac{\phi(x)}{T}}$$

since

$$\frac{1}{1+r} \geq C e^{-\delta r} \quad \text{for} \quad r \geq 0.$$

Thus

$$\begin{aligned}
\int_{\mathbf{R}^n} e^{(m(\lambda)-\delta)\frac{\phi(x)}{T}} e^{\frac{-\delta\phi(x)}{T}} (u_t^2 + |\nabla u|^2)dx &\leq CE_0 T' T^{2\delta - m(\lambda) - 1}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{T}} (u_t^2 + |\nabla u|^2)dx &\leq CE_0 T' T^{2\delta - m(\lambda) - 1}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} (u_t^2 + |\nabla u|^2)dx &\leq CE_0 (1+t)^{-\beta}(1+t)^{(1-\beta)(2\delta - m(\lambda) - 1)}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} (u_t^2 + |\nabla u|^2)dx &\leq CE_0 (1+t)^{(1-\beta)(2\delta - m(\lambda)) - 1}.
\end{aligned}$$

□

*Proof.* (**Proof of Corollary 1.3.2**) We add the estimates in Corollary (1.3.3), with a restriction on the integration to be on $\{x : \phi(x) \geq T^{1+\epsilon}\}$:

$$\int_{\phi(x) \geq T^{1+\epsilon}} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{T}} (u^2 + u_t^2 + |\nabla u|^2)dx \leq C(1+t)^c,$$

where $c < 0$ and depends on $\alpha$, $\beta$ and $n$. Since $\frac{\phi(x)}{T} \geq T^\epsilon \sim (1+t)^{\epsilon(1-\beta)}$, we have

$$\begin{aligned}
\int_{\phi(x) \geq T^{1+\epsilon}} (u^2 + u_t^2 + |\nabla u|^2)dx &\leq Ce^{-(m(\lambda)-2\delta)T^\epsilon} \\
&\leq Ce^{-(m(\lambda)-2\delta)(1+t)^{\epsilon(1-\beta)}}.
\end{aligned}$$

□

*Proof.* (**Proof of Corollary 1.3.3**) Substitute conditions $(A1)$ and $(A4)$ from Proposition (2.1.1) in the estimates of Theorem (1.3.1). □

# Chapter 3

# Nonlinear Wave Equations with Absorption $a(t, x)$

In this chapter we are studying the long time behavior of solutions to the nonlinear damped wave equation with space-time dependent potential. We will find decay estimates of the energy, $L^2$ and $L^{p+1}$ norm of the solutions.
Consider the following initial value problem:

$$u_{tt} - \Delta u + a(t,x)u_t + |u|^{p-1}u = 0, \quad x \in \mathbf{R}^n, \quad t > 0, \tag{3.0.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{3.0.2}$$

where $n \geq 3$ and $1 < p < (n+2)/(n-2)$. The initial data $(u_0, u_1)$ are compactly supported and belong to the energy space:

$$u_0 \in H^1(\mathbf{R}^n), \quad u_1 \in L^2(\mathbf{R}^n), \quad u_0(x) \text{ and } u_1(x) = 0 \text{ for } |x| > R.$$

The potential $a(t,x) = \lambda(x)\eta(t)$ is a $C^1(\mathbf{R}_+, \mathbf{R^n})$ function which is radially symmetric with respect to $x$. Under these conditions, using Theorem (1.1.7) problem (3.0.1)-(3.0.2) admits a global solution $u$ ( such that

$$u \in C((0,\infty), H^1(\mathbf{R}^n)),\ u_t \in C((0,\infty), L^2(\mathbf{R}^n)).$$

A classical result of Strauss [40] , Shatah and Stuwe [36] is the finite speed of propagation for (3.0.1)-(3.0.2), which yields

$$u(t,x) = 0 \text{ for } |x| > t + R.$$

We will extend the results of [45] concerning problem (1.3.11)-(1.3.12) to space-time dependent potentials $a(t,x) = \lambda(x)\eta(t)$, such that

**First case**

$$\lambda_0(1+|x|)^{-\alpha} \leq \quad \lambda(x) \quad \leq \lambda_1(1+|x|)^{-\alpha}, \quad \alpha \in [0,1), \tag{3.0.3}$$

$$\eta_0(1+t)^{-\beta} \leq \quad \eta(t) \quad \leq \eta_1(1+t)^{-\beta}, \quad \beta \in (-1,1); \tag{3.0.4}$$

For all $x \in \mathbf{R}^n$, $t \in \mathbf{R}_+$, $\lambda_0$, $\lambda_1$, $\eta_0$, $\eta_1 > 0$, such that $0 < \alpha + \beta < 1$.

**Second case**

$$\lambda_0(1+|x|)^{\alpha} \leq \quad \lambda(x) \quad \leq \lambda_1(1+|x|)^{\alpha}, \quad \alpha \in (0,\infty), \tag{3.0.5}$$

$$\eta_0(1+t)^{\beta} \leq \quad \eta(t) \quad \leq \eta_1(1+t)^{\beta}, \quad \beta \in (-1,1), \tag{3.0.6}$$

for all $(t,x) \in (\mathbf{R}_+, \mathbf{R}^n)$, $\lambda_0$, $\lambda_1$, $\eta_0$, $\eta_1 > 0$, such that

$$\frac{\eta_0^2}{\eta_1^2(1-\beta)^2} > \frac{\lambda_1}{3\lambda_0(n+\alpha)(2+\alpha)}. \tag{3.0.7}$$

We impose the following additional conditions on $\eta(t)$:

$$\begin{aligned} \left|\frac{d^k\eta(t)}{dt^k}\right| &\leq C_k\frac{\eta(t)}{(1+t)^k}, \qquad k\in\mathbf{N}, \\ \frac{d\eta(t)}{dt} &\leq 0. \end{aligned}$$

For all $t$ sufficiently large.

## 3.1 Deriving the Weighted Energy Identities

To find an approximate solution $w$ of (3.0.1), we rewrite (3.0.1) for separable damping terms $a(t,x)=\lambda(x)\eta(t)$:

$$u_{tt}-\Delta u+\lambda(x)\eta(t)u_t+|u|^{p-1}u=0,$$

where $\lambda(x),\ \eta(t)$ are positive $C^1$ coefficients. In fact, $w$ is allowed to be an approximate solution of the inequality

$$w_{tt}+\lambda(x)\eta(t)w_t-\Delta w\geq -f(t,x)w, \tag{3.1.1}$$

with a suitable function $f$ decaying sufficiently fast at infinity.

This equation is simpler when the potential is only space dependent. To be able to produce a space dependent potential we use the same transformation as in Section (2.1); hence $w$ will be an approximate solution of the following inequality:

$$w_{TT}+\lambda(x)w_T-\Delta w\geq -f(t,x)w, \tag{3.1.2}$$

where

$$T(t)=\int_0^t\frac{ds}{\eta(s)} \tag{3.1.3}$$

is such that $T \to \infty$ as $t \to \infty$; in fact, $T \sim (1+t)^{1-\beta}$, $T' = \frac{1}{\eta(t)} \sim (1+t)^{-\beta}$ as $t \to \infty$. The solution of inequality (3.1.2) can be chosen as

$$w(T,x) = T^{-m} e^{-\gamma \frac{\phi(x)}{T}}, \tag{3.1.4}$$

with suitable parameters $m$, $\gamma$. In the above $\phi(x)$ is a positive solution of the Poisson equation

$$\Delta\phi(x) = \lambda(x), \quad x \in \mathbf{R}^n, \tag{3.1.5}$$

with the following properties:

$$\begin{aligned}
&(a1) \qquad \phi(x) \geq 0 \ \text{ for all } \ x \in \mathbf{R}^n, \\
&(a2) \qquad \phi(x) = O(|x|^{2-\alpha}) \ \text{ for large } \ |x|, \\
&(a3) \qquad m(\lambda) = \liminf_{x\to\infty} \frac{\lambda(x)\phi(x)}{|\nabla\phi(x)|^2} \ > \ 0.
\end{aligned}$$

The choices of weights $w$ and function $f$ in (3.1.2) are delicate.

**Proposition 3.1.1.** *Let $\lambda(x)$ be a radially symmetric function in $C^1(\mathbf{R}^n)$, $n \geq 3$, which satisfies (3.0.3) Then equation (3.1.5) admits a solution $\phi \in C^2(\mathbf{R}^n)$, such that*

$$\begin{aligned}
&(A1) \qquad \phi_0(1+|x|)^{2-\alpha} \leq \phi(x) \leq \phi_1(1+|x|)^{2-\alpha}, \\
&(A2) \qquad m(\lambda) > 0,
\end{aligned}$$

*where $\phi_0$ and $\phi_1$ are positive constants. In the special case*

$$\lambda(x) \sim \lambda_2|x|^{-\alpha}, \quad |x| \to \infty, \tag{3.1.6}$$

*with $\lambda_2 > 0$, equation (3.1.5) has a solution with the following properties:*

$$(A3) \qquad \phi(x) \sim \frac{\lambda_2}{(2-\alpha)(n-\alpha)}|x|^{2-\alpha}, \quad |x| \to \infty,$$

$$(A4) \qquad m(\lambda) = \frac{n-\alpha}{2-\alpha}.$$

*Proof.* See Todorova and Yardanov [44]. □

To derive the modified equation for $v = w^{-1}u$ and the weighted energy identities, let us consider a general first-order perturbation of the semi-linear wave equation:

$$u_{tt} - \Delta u + au_t + b \cdot \nabla u + cu + |u|^{p-1}u = 0, \tag{3.1.7}$$

where the coefficients $a$, $b = (b_1, b_2, \ldots, b_n)$, $c \geq 0$ are in $C^1(\mathbf{R}_+, \mathbf{R}^n)$ functions. Using the transformation $v = w^{-1}u$, where $w$ is the approximate solution of (3.1.2), we have the following transformed equation:

$$v_{tt} - \Delta v + \hat{a}v_t + \hat{b} \cdot \nabla v + \hat{c}v + \hat{h}|v|^{p-1}v = 0. \tag{3.1.8}$$

Here the coefficients are given by

$$\begin{aligned}
&\hat{a} = a(t,x) + 2w^{-1}w_t, \\
&\hat{b} = b - 2w^{-1}\nabla w, \\
&\hat{c} = w^{-1}\Big(w_{tt} - \Delta w + a(t,x)w_t + b\nabla w + cw\Big), \\
&\hat{h} = w^{p-1}.
\end{aligned} \tag{3.1.9}$$

*Proof.* See Appendix $B$ with $k = 1$. □

To find the weighted energy identity, multiply the transformed equation (3.1.9) by $Pv_t + wv$. The weights $P,\ w \in C^2((0,\infty)\times \mathbf{R^n})$ are defined later.

**Proposition 3.1.2.** *Let $u \in C((0,\infty), H^2(\mathbf{R}^n)) \cap C^1((0,\infty), H^1(\mathbf{R}^n))$ be a solution of (3.1.7) with compactly supported data*

$$u_0 \in H^2(\mathbf{R}^n), \qquad u_1 \in H^1(\mathbf{R}^n)$$

*and source $\hat{h} \geq 0$. For any pair of $C^2$-functions $P$ and $w$, we have the equality*

$$\frac{d}{dt}E(v_t, \nabla v, v) + F(v_t, \nabla v) + G(v) + H(v) = 0, \tag{3.1.10}$$

*where*

$$\begin{aligned}
E(v_t,\nabla v, v) &= \frac{1}{2}\int_{\mathbf{R}^n}\Big(P(v_t^2 + |\nabla v|^2) + 2wv_tv + (\hat{c}P - w_t + \hat{a}w)v^2\Big)\ dx \\
&\quad + \frac{1}{p+1}\int_{\mathbf{R}^n} P\hat{h}|v|^{p+1}\ dx, \\
F(v_t, \nabla v) &= \frac{1}{2}\int_{\mathbf{R}^n}(-P_t + 2\hat{a}P - 2w)v_t^2\ dx \\
&\quad + \int_{\mathbf{R}^n}(\nabla P + \hat{b}P)\cdot v_t\nabla v\ dx \\
&\quad + \frac{1}{2}\int_{\mathbf{R}^n}(-P_t + 2w)|\nabla v|^2\ dx, \\
G(v) &= \frac{1}{2}\int_{\mathbf{R}^n}[w_{tt} - \Delta w - (\hat{a}w)_t - \nabla\cdot(w\hat{b}) + 2\hat{c}w - (\hat{c}P)_t]v^2\ dx, \\
H(v) &= \int_{\mathbf{R}^n}\left(w\hat{h} - \frac{1}{p+1}(P\hat{h})_t\right)|v|^{p+1}\ dx,
\end{aligned}$$

*with coefficients $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{h}$ given in (3.1.9).*

*Proof.* See Appendix $B$ with $k = 1$. $\square$

A simple consequence is that $E(t)$ is bounded if $F(t)+G(t)+H(t) \geq -g(t)$ with some $g \in L^1$. Next we relax the regularity conditions on $v$ showing that the same result holds for all $v$ in the energy space.

**Corollary 3.1.3.** *Let $u \in C((0,\infty), H^1(\mathbf{R}^n)) \cap C^1((0,\infty), L^2(\mathbf{R}^n))$ be a solution of equation (3.1.7) with compactly supported data and $h \geq 0$. Define the functionals $F$, $G$, $H$, and $E$ as in Proposition (3.1.2). If there exists a non-negative function $g \in L^1(\mathbf{R}_+)$, such that*

$$F(v_t, \nabla v) + G(v) + H(v) \geq -g(t), \quad t \geq t_0,$$

*then the weighted energy of $v$ satisfies*

$$E(v_t, \nabla v, v) \leq E_0 + \int_{t_0}^{t} g(s)\ ds, \quad t \geq t_0,$$

*where $E_0 = E(v_t, \nabla v, v)|_{t=t_0}$.*

*Proof.* If the data are compactly supported, such that $u_0 \in H^2$ and $u_1 \in H^1$, the identity (3.1.10) is satisfied. For any compactly supported data, such that $u_0 \in H^1$ and $u_1 \in L^2$, there exist compactly supported regular sequences $u_0^{(k)} \to u_0$ in $H^1$, $u_1^{(k)} \to u_1$ in $L^2$. Denote the corresponding solutions of (3.1.7) by $u^{(k)}$ and their weighted energy by $E(v_t, \nabla v, v)$. The weighted energy is controlled by the energy norm:

$$\begin{aligned} & |E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)}) - E(v_t, \nabla v, v)| \\ \leq\ & c(T^*) \left( \|u_t^{(k)} - u_t\|_2^2 + \|\nabla u^{(k)} - \nabla u\|_2^2 + \|u^{(k)} - u\|_{p+1}^{p+1} \right) \end{aligned}$$

whenever $t \in [0, T^*]$. and $\|u_t^{(k)}\|_2^2 + \|\nabla u^{(k)}\|_2^2 + \|u^{(k)}\|_{p+1}^{p+1} \leq M$, $\|u_t\|_2^2 + \|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1} \leq M$. The latter conditions hold with

$$M = c(T^*)(\|u_t(0,\cdot)\|_2^2 + \|\nabla u(0,\cdot)\|_2^2 + \|u(0,\cdot)\|_{p+1}^{p+1});$$

in fact $h \geq 0$ implies the following estimate for all $j$ and $t \in [0, T^*]$ :

$$\|v_t^{(j)}\|_2^2 + \|\nabla v^{(j)}\|_2^2 + \|v^{(j)}\|_{p+1}^{p+1} \leq c(T^*)(\|v_1^{(j)}\|_2^2 + \|\nabla v_0^{(j)}\|_2 + \|v_0^{(j)}\|_{p+1}^{p+1}).$$

The energy norm of solutions of problem (3.0.1) is a locally Lipschitz function of the initial data if $p \in [1, (n+2)/(n-2))$, see [36] or [40]. Hence

$$\begin{aligned} & |E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)}) - E(v_t, \nabla v, v)| \\ \leq \; & c(T^*) \left( \|u_1^{(k)} - u_1\|_2^2 + \|\nabla u_0^{(k)} - \nabla u_0\|_2^2 + \|u_0^{(k)} - u_0\|_{p+1}^{p+1} \right), \end{aligned}$$

which yields

$$\begin{aligned} E(v_t, \nabla v, v) &= \lim_{k \to \infty} E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)}) \\ &\leq \lim_{k \to \infty} E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)})|_{t=t_0} + \int_{t_0}^{t} g(s)\; ds \\ &= E_0 + \int_{t_0}^{t} g(s)\; ds. \end{aligned}$$

□

Comparing the damped wave equation (3.0.1) with equation (3.1.7), we have

$$a(t,x) = \eta(t)\lambda(x), \quad b = 0, \quad \text{and} \quad c = 0$$

so the transformed coefficients will be

$$
\begin{aligned}
\hat{a} &= \eta(t)\lambda(x) + 2w^{-1}w_t, \\
\hat{b} &= -2w^{-1}\nabla w, \\
\hat{c} &= w^{-1}(w_{tt} - \Delta w + \eta(t)\lambda(x)w_t), \\
\hat{h} &= w^{p-1}.
\end{aligned}
\tag{3.1.11}
$$

The weighted identity for $v$ in Proposition (3.1.2) can be simplified if we use the transformed coefficients as in (3.1.11). For simplicity we keep the complex coefficient $\hat{c}$, hence we have the following result.

**Proposition 3.1.4.** *Let $u$ be the solution of (3.0.1) with compactly supported data in the energy space and define $v = w^{-1}u$, where $w > 0$ is a $C^2$-function. Then*

$$
\begin{aligned}
E(v_t, \nabla v, v) &= \frac{1}{2}\int_{\mathbf{R}^n} \Big(P(v_t^2 + |\nabla v|^2) + 2wv_tv + (\hat{c}P + w_t + aw)v^2\Big)\,dx \\
&\quad + \frac{1}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\,dx,
\end{aligned}
$$

$$
\begin{aligned}
F(v_t, \nabla v) &= \frac{1}{2}\int_{\mathbf{R}^n} (-P_t + 2aP + 4P(\ln w)_t - 2w)v_t^2\,dx \\
&\quad + \int_{\mathbf{R}^n} (\nabla P - 2P\nabla \ln w)\cdot v_t\nabla v\,dx \\
&\quad + \frac{1}{2}\int_{\mathbf{R}^n} (-P_t + 2w)|\nabla v|^2\,dx,
\end{aligned}
$$

$$
G(v) = \frac{1}{2}\int_{\mathbf{R}^n} [\hat{c}w - (\hat{c}P)_t]v^2\,dx,
$$

$$
H(v) = \int_{\mathbf{R}^n} \left(w^p - \frac{1}{p+1}(Pw^{p-1})_t\right)|v|^{p+1}\,dx.
$$

*Moreover, the condition $F(v_t, \nabla v) + G(v) + H(v) \geq -g(t)$ with a non-negative function $g \in L^1(\mathbf{R}_+)$ implies that the weighted energy of $v$ satisfies*

$$E(v_t, \nabla v, v) \leq E_0 + \int_{t_0}^{t} g(s)\ ds, \quad t \geq t_0,$$

*where $E_0 = E(v_t, \nabla v, v)|_{t=t_0}$.*

*Proof.* We combine the definitions of $E(v_t, \nabla v), v)$, $F(v_t, \nabla v)$, $G(v)$ and $H(v)$ in Proposition (3.1.2) and the coefficients (3.1.11). $\square$

## 3.2 Sufficient Conditions on the Weights

In this section we show that Proposition (3.1.4) applies to all positive weights $P$ and $w$ satisfying different inequalities. Hence the weighted energy $E(v_t, \nabla v, v)$ is a bounded function of time for such weights. Another problem is to show that the weighted energy is positive definite, so it can be used for decay estimates of $v$ and the original solution $u = wv$.

**Proposition 3.2.1.** *Assume that $P > 0$ and $w > 0$ are $C^1$-functions, such that*

$$\begin{aligned}
&(i) && \hat{c}w - (\hat{c}P)_t \geq C^- w, \quad \textit{where} \quad \int_{t_0}^{\infty} \int_{\mathbf{R}^n} w^{-1} |C^-|^{\frac{p+1}{p-1}}\ dxdt < \infty,\\
&(ii) && -P_t + w \geq 0,\\
&(iii) && (-P_t + 2w)(-P_t + 2aP + 4P(\ln w)_t - 2w) \geq (\nabla P - 2P\nabla \ln w)^2,\\
&(iv) && (p+1)w^p - (Pw^{p-1})_t \geq w^p,
\end{aligned}$$

*for all $t \geq t_0$ sufficiently large, $|x| \leq t + R$. Let $u$ be a solution of (3.0.1) in the energy space and define $v = w^{-1}u$. Then*

$$F(v_t, \nabla v) + G(v) + H(v) \geq -c \int_{\mathbf{R}^n} w^{-1} |C^-|^{\frac{p+1}{p-1}}\ dx;$$

*see the definitions of these functionals in Proposition (3.1.4). Thus the weighted energy* $E(v_t, \nabla v, v)$ *satisfies*

$$\begin{aligned} E_0 + C_0 \;\; &\geq \;\; \frac{1}{2}\int_{\mathbf{R}^n} [P(v_t^2 + |\nabla v|^2) + 2wv_tv + (\hat{c}P + w_t + aw)v^2]\; dx \\ &+ \frac{1}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\; dx, \end{aligned} \tag{3.2.1}$$

*for all* $t \geq t_0$, *where*

$$E_0 = E(v_t, \nabla v, v)|_{t=t_0}, \quad C_0 = C\int_{t_0}^{\infty}\int_{\mathbf{R}^n} w^{-1}|C^-|^{\frac{p+1}{p-1}}\; dxdt.$$

*Proof.* Conditions $(ii)$ and $(iii)$ insure that $F(v_t, \nabla v) \geq 0$ as a non-negative quadratic form of $v_t$ and $\nabla v$. Using condition $(i)$ together with Young's inequality with exponents $\frac{p+1}{2}$ and $\frac{p+1}{p-1}$ a lower bound on $G(v)$ will be given as follows:

$$\begin{aligned} G(v) \;\; &= \;\; \frac{1}{2}\int_{\mathbf{R}^n} [\hat{c}w - (\hat{c}P)_t]v^2\; dx \\ &\geq \;\; \frac{1}{2}\int_{\mathbf{R}^n} C^- wv^2\; dx \\ &\geq \;\; -\int_{\mathbf{R}^n} w^p|v|^{p+1}\; dx - C\int_{\mathbf{R}^n} w^{-1}|C^-|^{\frac{p+1}{p-1}}\; dxdt. \end{aligned}$$

Condition $(iv)$ yields

$$H(v) \;\; \geq \;\; \frac{1}{p+1}\int_{\mathbf{R}^n} w^p|v|^{p+1}\; dx,$$

so we have

$$G(v) + H(v) \;\; \geq \;\; -C\int_{\mathbf{R}^n} w^{-1}|C^-|^{\frac{p+1}{p-1}}\; dxdt.$$

Now apply Proposition (3.1.4) to complete the proof. □

## 3.3 Case one: $\lambda(x)$ and $\eta(t)$ are given by (3.0.3) and (3.0.4)

We need an upper bound on the indefinite term $2wvv_t$ so that the weighted energy $E(v_t, \nabla v, v)$ will be positive definite. To guarantee that $\hat{c}Pv^2$ is dominated by the positive nonlinearity $Pw^{p-1}|v|^{p+1}$ new conditions on $P$ and $w$ need to be identified.

**Proposition 3.3.1.** *Assume that conditions (i)-(iv) in Proposition (3.2.1) hold for $P$ and $w$ and $a(t,x) = \lambda(x)\eta(t)$, where $\lambda(x)$ and $\eta(t)$ are defined by (3.0.3) and (3.0.4) respectively. If*

$$(v) \qquad \hat{c} \geq C^- \quad \textit{and} \quad \sup_{t \geq t_0} \int_{\mathbf{R}^n} Pw^{-2}|C^-|^{\frac{p+1}{p-1}}\ dx < \infty,$$

*then*

$$\int_{\mathbf{R}^n} wv^2\ dx \leq N_0 + C(E_0+1)t^{\alpha+\beta},$$

*for $t \geq t_0$, where $N_0 = \int_{\mathbf{R}^n} wv^2\ dx\ |_{t=t_0}$ .*

*Proof.* Drop $\int_{\mathbf{R}^n} P(v_t^2 + |\nabla u|^2)\ dx$ in inequality (3.2.1) from Proposition (3.2.1) and use $\frac{d}{dt}(wv)^2 = 2wv_tv + w_tv^2$. Then

$$\begin{aligned}
& \frac{d}{dt}\int_{\mathbf{R}^n} wv^2\ dx + \int_{\mathbf{R}^n} awv^2\ dx \\
\leq\ & 2(E_0 + C_0) - \int_{\mathbf{R}^n} \hat{c}Pv^2\ dx - \frac{2}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\ dx, \\
\leq\ & 2(E_0 + C_0) - \int_{\mathbf{R}^n} C^-Pv^2\ dx - \frac{2}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\ dx, \\
\leq\ & 2(E_0 + C_0) + \int_{\mathbf{R}^n} |C^-|Pv^2\ dx - \frac{2}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\ dx.
\end{aligned}$$

To estimate $|C^-|Pv^2$, we apply Young's inequality with exponents $\frac{p+1}{2}$ and $\frac{p+1}{p-1}$:

$$|C^-|Pv^2 \le \frac{2}{p+1} Pw^{p-1}|v|^{p+1} + CPw^{-2}|C^-|^{\frac{p+1}{p-1}}.$$

Hence

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbf{R}^n} wv^2\, dx + \int_{\mathbf{R}^n} awv^2\, dx \;&\le\; 2(E_0+C_0) + \frac{2}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\, dx \\
&+\; C\int_{\mathbf{R}^n} Pw^{-2}|C^-|^{\frac{p+1}{p-1}}\, dx - \frac{2}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\, dx.
\end{aligned}$$

Define

$$b_0 = C\sup_{t\ge t_0}\int_{\mathbf{R}^n} Pw^{-2}|C^-|^{\frac{p+1}{p-1}}\, dx.$$

Thus we obtain

$$\frac{d}{dt}\int_{\mathbf{R}^n} wv^2\, dx + \int_{\mathbf{R}^n} awv^2\, dx \le 2(E_0+C_0) + b_0, \quad t \ge t_0.$$

We can now estimate the weighted $L^2$ norm of $v$. Using $|x| \le t+R$ and $a(t,x) = \lambda(x)\eta(t)$, where $\lambda(x)$ and $\eta(t)$ are defined by (3.0.3) and (3.0.4) respectively,

$$\frac{d}{dt}\int_{\mathbf{R}^n} wv^2\, dx + \frac{\lambda_0\eta_0}{(1+t+R)^{\alpha+\beta}}\int_{\mathbf{R}^n} wv^2\, dx \le C(E_0+1).$$

This inequality is equivalent to

$$\frac{d}{dt}\left(e^{\frac{\lambda_0\eta_0}{1-\alpha-\beta}(1+t+R)^{1-\alpha-\beta}}\int_{\mathbf{R}^n} wv^2\, dx\right) \le C(E_0+1)e^{\frac{\lambda_0\eta_0}{1-\alpha-\beta}(1+t+R)^{1-\alpha-\beta}}.$$

Integrating on $[t_0, t]$,

$$\int_{t_0}^{t}\frac{d}{dt}\left(e^{\frac{\lambda_0\eta_0}{1-\alpha-\beta}(1+s+R)^{1-\alpha-\beta}}\int_{\mathbf{R}^n} wv^2\, dx\right)\, ds \le C(E_0+1)\int_{t_0}^{t} e^{\frac{\lambda_0\eta_0}{1-\alpha-\beta}(1+s+R)^{1-\alpha-\beta}}\, ds.$$

We need the following estimate:

$$\int_{t_0}^{t} e^{c\tau^{1-\rho}} d\tau = \frac{1}{1-\rho} \int_{t_0^{1-\rho}}^{t^{1-\rho}} e^{cz} z^{\frac{\rho}{1-\rho}} dz \le \frac{t^{\rho} e^{ct^{1-\rho}}}{c(1-\rho)}$$

with $c = \frac{\lambda_0 \eta_0}{1-\alpha-\beta}$, $\tau = 1+s+R$ and $\rho = \alpha+\beta$. Hence

$$\begin{aligned} C(E_0+1)\int_{t_0}^{t} e^{\frac{\lambda_0\eta_0}{1-\alpha-\beta}(1+s+R)^{1-\alpha-\beta}} \, ds &\le C(E_0+1)\frac{t^{\alpha+\beta} e^{\frac{\lambda_0\eta_0}{1-\alpha-\beta} t^{1-\alpha-\beta}}}{\frac{\lambda_0\eta_0}{1-\alpha-\beta}(1-\alpha-\beta)}, \\ &= C(E_0+1) t^{\alpha+\beta} e^{\frac{\lambda_0\eta_0}{1-\alpha-\beta} t^{1-\alpha-\beta}}. \end{aligned}$$

Finally,

$$\begin{aligned} e^{\frac{\lambda_0\eta_0}{1-\alpha-\beta}(1+t+R)^{1-\alpha-\beta}} \int_{\mathbf{R}^n} wv^2 \, dx &\le M_0 + C(E_0+1) t^{\alpha+\beta} e^{\frac{\lambda_0\eta_0}{1-\alpha-\beta} t^{1-\alpha-\beta}}, \\ \int_{\mathbf{R}^n} wv^2 \, dx &\le N_0 + C(E_0+1) t^{\alpha+\beta}. \end{aligned}$$

□

**Proposition 3.3.2.** *Assume that $a(t,x) = \lambda(x)\eta(t)$, where $\lambda(x)$ and $\eta(t)$ are defined by (3.0.3) and (3.0.4) respectively. If the weights $P$ and $w$ satisfy conditions (i)–(iv) in Proposition (3.2.1), condition (v) in Proposition (3.3.1), and*

$$(vi) \qquad w \le C t^{-(\alpha+\beta)} P,$$

$$(vii) \qquad w_t \ge C t^{-(\alpha+\beta)} w,$$

*we have*

$$\begin{aligned}\int_{\mathbf{R}^n} P(v_t^2+|\nabla v|^2)\ dx &\leq C(N_0+E_0+1),\\ \int_{\mathbf{R}^n} awv^2\ dx &\leq C(N_0+E_0+1),\\ \int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\ dx &\leq C(N_0+E_0+1),\end{aligned}$$

*for all $t \geq t_0$.*

*Proof.* Using the result of Proposition (3.3.1)

$$\begin{aligned}2(E_0+C_0) \geq & \int_{\mathbf{R}^n} [P(v_t^2+|\nabla v|^2)+2wv_tv+(\hat{c}P+w_t+aw)v^2]\ dx\\ &+\frac{2}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\ dx.\end{aligned}$$

This result and $|2wv_tv| \leq \epsilon t^{\alpha+\beta}wv_t^2+\epsilon^{-1}t^{-(\alpha+\beta)}wv^2$, where $\epsilon \in (0,1)$, imply

$$\begin{aligned}&\int_{\mathbf{R}^n}(P-\epsilon t^{\alpha+\beta}w)(v_t^2+|\nabla v|^2)\ dx+\int_{\mathbf{R}^n} awv^2\ dx\\ \leq\ & 2(E_0+C_0)+\int_{\mathbf{R}^n}(\epsilon^{-1}t^{-(\alpha+\beta)}w-w_t)v^2\ dx\\ &+\int_{\mathbf{R}^n}|C^-|Pv^2\ dx-\frac{2}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\ dx.\end{aligned}$$

Applying similar argument as of Proposition (3.3.1), we have that

$$\begin{aligned}&\int_{\mathbf{R}^n}(P-\epsilon t^{(\alpha+\beta)}w)(v_t^2+|\nabla v|^2)\ dx+\int_{\mathbf{R}^n} awv^2\ dx+\frac{1}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\ dx\\ \leq\ & 2(E_0+C_0)+b_0+\int_{\mathbf{R}^n}(\epsilon^{-1}t^{-(\alpha+\beta)}w-w_t)v^2\ dx. \qquad (3.3.1)\end{aligned}$$

Using conditions $(iv)$ and $(v)$ equation (3.3.1) will be

$$\begin{aligned}
& (1-\epsilon C_1)\int_{\mathbf{R}^n} P(v_t^2+|\nabla v|^2)\ dx + \int_{\mathbf{R}^n} awv^2\ dx + \frac{1}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\ dx \\
\leq\ & 2(E_0+C_0)+b_0+(C_1+\epsilon^{-1})t^{-(\alpha+\beta)}\int_{\mathbf{R}^n} wv^2\ dx
\end{aligned}$$

Choose $\epsilon = (2C_1)^{-1}$

$$\begin{aligned}
& \frac{1}{2}\int_{\mathbf{R}^n} P(v_t^2+|\nabla v|^2)\ dx + \int_{\mathbf{R}^n} awv^2\ dx + \frac{1}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\ dx \\
\leq\ & 2(E_0+C_0)+b_0+Ct^{-(\alpha+\beta)}\int_{\mathbf{R}^n} wv^2\ dx,
\end{aligned}$$

using the result of Proposition (3.3.1), hence we have the given inequalities. $\square$

Restate Proposition (3.3.1) and Proposition (3.3.2) in terms of $u$. Using $v = w^{-1}u$ we readily obtain

$$\begin{aligned}
\int_{\mathbf{R}^n} w^{-1}u^2\ dx &\leq N_0 + C(E_0+1)t^{\alpha+\beta}, \\
\int_{\mathbf{R}^n} aw^{-1}u^2\ dx &\leq C(N_0+E_0+1), \\
\int_{\mathbf{R}^n} Pw^{-2}|u|^{p+1}\ dx &\leq C(N_0+E_0+1).
\end{aligned}$$

**Proposition 3.3.3.** *Assume that $a(t,x) = \lambda(x)\eta(t)$, where $\lambda(x)$ and $\eta(t)$ are defined by (3.0.3) and (3.0.4) respectively, $P$ and $w$ satisfy (i)-(vii) and*

$$(viii) \qquad Pw^{-3}(w_t^2+|\nabla w|^2) \leq Ca(t,x),$$

*where $C$ is a constant. Then the solution $u$ of (3.0.1) satisfies*

$$\begin{aligned}
\int_{\mathbf{R}^n} a w^{-1} u^2 \, dx &\leq C(N_0 + E_0 + 1), \\
\int_{\mathbf{R}^n} P w^{-2} (u_t^2 + |\nabla u|^2) \, dx &\leq C(N_0 + E_0 + 1), \\
\int_{\mathbf{R}^n} P w^{-2} |u|^{p+1} \, dx &\leq C(N_0 + E_0 + 1),
\end{aligned}$$

*for $t \geq t_0$. Here $C$ depends only on the equation and weights.*

*Proof.* To show the second estimate, use $v = w^{-1} u$ so

$$v_t^2 = (-w^{-2} w_t u + w^{-1} u_t)^2 \geq \frac{1}{2} w^{-2} u_t^2 - 3 w^{-4} w_t^2 u^2$$

and

$$|\nabla v|^2 = (-w^{-2} \nabla w u + w^{-1} \nabla u)^2 \geq \frac{1}{2} w^{-2} |\nabla u|^2 - 3 w^{-4} |\nabla w|^2 u^2.$$

Thus

$$\frac{1}{2} P w^{-2} (u_t^2 + |\nabla u|^2) \leq P(v_t^2 + |\nabla u|^2) + 3 P w^{-4} (w_t^2 + |\nabla w|^2) u^2.$$

Integrate this inequality and apply Proposition (3.3.2), we have

$$\frac{1}{2} \int_{\mathbf{R}^n} P w^{-2} (u_t^2 + |\nabla u|^2) \, dx \leq C(N_0 + E_0 + 1) + 3 \int_{\mathbf{R}^n} P w^{-4} (w_t^2 + |\nabla w|^2) u^2 \, dx.$$

Applying condition $(v)$, $P w^{-4} (w_t^2 + |\nabla w|^2) \leq C_2 a(t, x) w^{-1}$ and Proposition (3.3.1) to the right side, we have the needed estimate. □

The above result yields non-trivial estimates if there exist weights with properties $(i)$ – $(viii)$ which decay sufficiently fast as $t$ and $|x|$ go to infinity. We found a new pair of weights to accommodate the existence of the nonlinear term in problem (3.0.1). Both weights are defined in terms of certain positive solutions to the Poisson equation in $\mathbf{R}^n$. More details about the weights are in the next section.

### 3.3.1 Construction of Weights

In this section we define a family of weights $P$ and $w$ satisfying conditions $(i)-(viii)$. We rely on the results from [44], [45] and Proposition (3.1.1). Given the parameters $S_0 > 0$ and $\gamma > 0$, we introduce

$$S(x) = \gamma\phi(x) + S_0, \quad x \in \mathbf{R^n},$$

where $\phi(x)$ is the function defined in Proposition (3.1.1). We are using the same weights as in the previous chapter:

$$w(T,x) = T^{-m}e^{-\frac{S(x)}{T}}, \quad P(T,x) = \frac{3}{4}\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1}\frac{w(T,x)}{T'}. \tag{3.3.2}$$

Note that $S$, $P$, $w > 0$ on $\mathbf{R}_+ \times \mathbf{R}^n$. $T$ is defined above (3.1.3), and $m > 0$ is an additional parameter. The two numbers $\gamma$ and $m$ depend on $n$, $p$ and $a$. We need to show the additional conditions that have been introduced due to the existence of absorption term.

$(ii)$ To show $-P_t + w \geq 0$, we use Lemma (A.4) and $T \sim t^{\beta+1}$. We have

$$\begin{aligned}
\left(-\frac{P_t}{P} + \frac{w}{P}\right)P &\geq t^\beta P\left(\frac{m-1}{t^{\beta+1}} + \frac{t^{\beta-1}}{t^{2\beta}} + \frac{8}{t^{\beta+1}}\right)\\
&= P\left(\frac{m-1}{t} + \frac{1}{t} + \frac{8}{t}\right)\\
&= P\left(\frac{m+8}{t}\right) \geq 0.
\end{aligned}$$

$(iii)$ The next condition is

$$\left(-P_t + 2a(t,x)P + 4P(\ln w)_t - 2w\right)\left(-P_t + 2w\right) \geq \left(\nabla P - 2P\nabla \ln w\right)^2.$$

Using Lemma (A.4), the first term of the left hand side is

$$\begin{aligned}
-P_t + 2aP + 4P(\ln w)_t - 2w &\geq \bigg( -\frac{(3m+17)t^\beta}{t^{\beta+1}} + \frac{t^{\beta-1}}{t^\beta} + 2\lambda(x)\eta(t)\bigg)P \\
&= \bigg( -\frac{3m+17}{t} + \frac{1}{t} + 2\lambda(x)\eta(t)\bigg)P \\
&= \bigg( -\frac{3m+16}{t} + 2\lambda(x)\eta(t)\bigg)P \\
&\geq a(t,x)P.
\end{aligned}$$

The rest of the proof of $(iii)$ is the same as in Chapter 2.

$(iv)$ We need $(p+1)w^p - (Pw^{p-1})_t \geq w^p$ or $pw^p - (Pw^{p-1})_t \geq 0$.

To show this we use condition $(ii)$ and two simple computations:

$$-P_t + w \geq 0, \quad \frac{w_t}{w} = \bigg( -\frac{m}{T} + \frac{S(x)}{T^2}\bigg)T', \quad \frac{P}{w} = \frac{3}{4}\bigg(\frac{6}{T} + \frac{S(x)}{T^2}\bigg)^{-1}\frac{1}{T'}.$$

Hence

$$-\frac{P_t}{w} \geq -1, \quad \frac{P}{w}\frac{w_t}{w} \leq \frac{3}{4}.$$

We now obtain

$$\begin{aligned}
pw^p - (Pw^{p-1})_t &= pw^p - P_t w^{p-1} - P(p-1)w^{p-2}w_t \\
&= w^p\bigg(p - \frac{P_t}{w} - (p-1)\frac{P}{w}\frac{w_t}{w}\bigg) \\
&\geq w^p\bigg(p - 1 - \frac{3}{4}(p-1)\bigg) \\
&= \frac{1}{4}w^p(p-1) \geq 0.
\end{aligned}$$

$(vi)$ It is easy to check that $w \leq Ct^{-(\alpha+\beta)}P$. In fact,

$$\begin{aligned} \frac{w}{P} &= \frac{4}{3}\left(\frac{6T+S(x)}{T^2}\right)T' \\ &\leq \frac{4}{3}\left(\frac{6t^{1+\beta}+t^{2-\alpha}}{t^{2\beta+2}}\right)t^{\beta}. \end{aligned}$$

Since $t^{1+\beta} \leq Ct^{2-\alpha}$ we have

$$\frac{w}{P} \leq Ct^{-(\alpha+\beta)}.$$

$(vii)$ To verify $w_t \geq Ct^{-(\alpha+\beta)}w$, we calculate

$$\begin{aligned} \frac{w_t}{w} &= \left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)T' \\ &\geq -\frac{mT'}{T} \\ &\geq -\frac{mt^{\beta}}{t^{1+\beta}} \\ &= -\frac{m}{t}. \end{aligned}$$

Since $\alpha+\beta<1$, we obtain $t^{\alpha+\beta}<t$ and

$$\frac{w_t}{w} \geq -\frac{m}{t^{\alpha+\beta}}.$$

$(viii)$ The last condition is $Pw^{-3}\left(w_t^2+|\nabla w|^2\right) \leq Ca(t,x)$.

Using Lemma (A.4),

$$\begin{aligned} Pw^{-3}\left(w_t^2+|\nabla w|^2\right) &\leq C\left(\frac{m^2t^{\beta}}{t^{\beta+1}}+\frac{t^{2-\alpha}}{t^{2\beta+2}}t^{\beta}+\lambda(x)\eta(t)\right) \\ &\leq C\left(\frac{m^2}{t}+\frac{1}{t^{\alpha+\beta}}+\lambda(x)\eta(t)\right) \\ &\leq Ca(t,x). \end{aligned}$$

The remaining two conditions $(i)$ and $(v)$ are related to the asymptotic behavior of $w$ as a solution to

$$w_{TT} - \Delta w + \lambda(x) w_T \geq C^- w. \tag{3.3.3}$$

We choose $C^-$ so that $m$ is maximized. The choice of $C^-$ is different if the exponent $p$ of the nonlinear term is large or small. The parameters $m$ and $\gamma$ have to be chosen such that the remanning two conditions $(i)$ and $(v)$ are satisfied.

*The choice of the parameters $m$ and $\gamma$ for the supercritical exponents $p$.*

**Lemma 3.3.4.** *Let $m$ and $S$ be defined as follows:*

$$m = m(\lambda) - 2\delta, \quad S(x) = (m(\lambda) - \delta)\phi(x) + S_0,$$

*where $\delta \in (0, \frac{1}{2}m(\lambda))$, $S_0 > 0$, $m(\lambda)$ and $\phi(x)$ has properties $(a1)$-$(a3)$. Then*

$$\begin{aligned} &(S1) \qquad \Delta S(x) = (m + \delta)\lambda(x) \text{ for all } x \in \mathbf{R}^n, \\ &(S2) \qquad S(x) = O(|x|^{2-\alpha}), \\ &(S3) \qquad \left(1 - \frac{\delta}{2m(\lambda)}\right) \lambda(x) S(x) \geq |\nabla S(x)|^2 \text{ for all } x \in \mathbf{R}^n. \end{aligned}$$

*Proof.* See Todorova and Yordanov [44]. □

**Proposition 3.3.5.** *Let $P$ and $w$ be defined in (3.3.2) then*

$$\hat{c} \geq 0, \quad \hat{c}_t \leq 0.$$

*Proof.* Using Lemma (A.3) and conditions $(S1)$ and $(S3)$ of Lemma (3.3.4), we have

$$\begin{aligned}
\hat{c} &\geq \frac{-mT'' + (m+\delta)\lambda(x) - m\lambda(x)}{T} \\
&+ \frac{S(x)T'' - (1-k\delta)\lambda(x)S(x) + \lambda(x)S(x) - \frac{2S(x)T'^2}{T}}{T^2} \\
&= \frac{-mT'' + \delta\lambda(x)}{T} + \frac{S(x)\left(T'' + k\delta\lambda(x) - \frac{2T'^2}{T}\right)}{T^2}.
\end{aligned}$$

From the finite speed of propagation, $1 + |x| \leq 1 + t + R$ and

$$\lambda(x) \geq \lambda_0(1+|x|)^{-\alpha} \geq (1+t)^{-\alpha}.$$

Hence

$$\begin{aligned}
\hat{c} &\geq \frac{-mT'' + \delta(1+t)^{-\alpha}}{T} + \frac{S(x)\left(T'' + k\delta(1+t)^{-\alpha} - \frac{2T'^2}{T}\right)}{T^2} \\
&\geq \frac{-m(1+t)^{\beta-1} + \delta(1+t)^{-\alpha}}{(1+t)^{\beta+1}} + \frac{S(x)}{(1+t)^{2\beta+2}}\left((1+t)^{\beta-1} + k\delta(1+t)^{-\alpha} - \frac{2(1+t)^{2\beta}}{(1+t)^{\beta+1}}\right) \\
&= \frac{-m}{(1+t)^2} + \frac{\delta}{(1+t)^{\alpha+\beta+1}} + S(x)\left((1+t)^{-\beta-3} + \frac{k\delta}{(1+t)^{\alpha+2\beta+2}} - 2(1+t)^{-\beta-3}\right) \\
&= \frac{-m}{(1+t)^2} + \frac{\delta}{(1+t)^{\alpha+\beta+1}} + S(x)\left(\frac{-1}{(1+t)^{\beta+3}} + \frac{k\delta}{(1+t)^{\alpha+2\beta+2}}\right).
\end{aligned}$$

We see that $\hat{c} \geq 0$ for sufficiently large time $t$, whenever $\alpha + \beta < 1$. The proof of $\hat{c}_t \leq 0$ is similar. □

**Proposition 3.3.6.** *Let $P$ and $w$ be defined in (3.3.2) and $m$ and $\gamma$ be defined as follows:*

$$m = m(\lambda) - 2\delta, \quad \gamma = m(\lambda) - \delta. \tag{3.3.4}$$

*Here $m(\lambda)$ is given by $(a3)$ and $\delta > 0$ is a small number. Then conditions $(i)$ and $(v)$ hold for sufficiently large $t \geq t_0$.*

*Proof.* As a consequence of Proposition (3.3.5) we can take $C^- = 0$. Hence the integrals in conditions $(i)$ and $(v)$ are bounded. Moreover,

$$\hat{c}w - (\hat{c}P)_t = \hat{c}(w - P_t) - \hat{c}_t P \geq 0.$$

Thus the inequality in condition $(i)$ is satisfied. $\square$

*The choice of the parameters $m$ and $\gamma$ for the* ***subcritical*** *exponents $p$.*

**Proposition 3.3.7.** *Let $P$ and $w$ be defined in (3.3.2) and $m$ and $\gamma$ be defined as follows:*

$$\begin{aligned} m &= -\frac{1}{1+\beta} + \frac{p+1}{p-1} + \min\left\{\frac{p+1}{p-1}\frac{\alpha}{2-\alpha} - \frac{n}{2-\alpha}, 0\right\} - \delta, \\ \gamma &= m(\lambda) - \delta. \end{aligned} \tag{3.3.5}$$

*Then conditions $(i)$ and $(v)$ hold for sufficiently large $t \geq t_0$.*

Before proving Proposition (3.3.7) we need to find a lower bound for $\hat{c}$ and an upper bound for $t\hat{c}_t$.

**Lemma 3.3.8.** *There exist positive constants $k_i$, $i = 1, 2, 3, 4$, such that*

$$\begin{aligned} \hat{c}(t,x) &\geq -k_1(1+|x|)^{-\alpha}t^{-(\beta+1)} + k_2(1+|x|)^{2-2\alpha}t^{-2(\beta+1)}, \\ t\hat{c}_t(t,x) &\leq k_3(1+|x|)^{-\alpha}t^{-(\beta+1)} - k_4(1+|x|)^{2-2\alpha}t^{-2(\beta+1)}. \end{aligned}$$

*Proof.* Using inequality (A.4) from Lemma (A.3),

$$\begin{aligned}
\hat{c} \;\; &\geq \;\; \left(-\frac{m}{T} + \frac{S(x)}{T^2}\right) T'' + \left(\frac{m}{T^2} - \frac{2S(x)}{T^3}\right) T'^2 + \frac{\lambda(x)S(x) - |\nabla S(x)|^2}{T^2} + \\
&+ \;\; \frac{\Delta S(x) - m\lambda(x)}{T},
\end{aligned}$$

where $S(x) = \gamma\phi(x) + S_0$, $\gamma$ is defined by (3.3.5), and $\phi(x)$ is a positive solution of the Poisson equation. The main difference between this proposition and the previous one is that $m$ can be larger than $m(\lambda)$. From $(S1)$ of Lemma (3.3.4), we obtain

$$\begin{aligned}
\Delta S(x) - m\lambda(x) &= (m(\lambda) - \delta)\triangle\phi(x) - m\lambda(x) \\
&= (m(\lambda) - \delta)\lambda(x) - m\lambda(x) \\
&= (m(\lambda) - m - \delta)\lambda(x) \\
&\geq -k_1(1 + |x|)^{-\alpha}.
\end{aligned}$$

However, by condition $(S3)$ of Lemma (3.3.4),

$$\begin{aligned}
\lambda(x)S(x) - |\nabla S(x)|^2 &\geq \lambda(x)S(x) - \left(1 - \frac{\delta}{2m(\lambda)}\right)\lambda(x)S(x) \\
&= \frac{\delta}{2m(\lambda)}\lambda(x)S(x) \\
&\geq k_2(1 + |x|)^{-\alpha}(1 + |x|)^{2-\alpha} \\
&= k_2(1 + |x|)^{2-2\alpha}, \text{ where } k_2 > 0.
\end{aligned}$$

Hence

$$
\begin{aligned}
\hat{c} \;\geq\;& -k_1 t^{-\beta-1}(1+|x|)^{-\alpha} + k_2 t^{-2\beta-2}(1+|x|)^{2-2\alpha} - \frac{mt^{\beta-1}}{t^{\beta+1}} + \frac{S(x)t^{\beta-1}}{t^{2\beta+2}} \\
+\;& \frac{mt^{2\beta}}{t^{2\beta+2}} - \frac{2S(x)t^{2\beta}}{t^{3\beta+3}} \\
=\;& -k_1 t^{-\beta-1}(1+|x|)^{-\alpha} + k_2 t^{-2\beta-2}(1+|x|)^{2-2\alpha} + S(x)t^{-\beta-3} - 2S(x)t^{-\beta-3} \\
=\;& -k_1 t^{-\beta-1}(1+|x|)^{-\alpha} + k_2 t^{-2\beta-2}(1+|x|)^{2-2\alpha} - S(x)t^{-\beta-3} \\
\geq\;& -k_1 t^{-\beta-1}(1+|x|)^{-\alpha} + k_2 t^{-2\beta-2}(1+|x|)^{2-2\alpha},
\end{aligned}
$$

for sufficiently large $t \geq t_0$.

The estimate of $t\hat{c}_t$ is very similar, since $\hat{c}$ is a polynomial of $t^{-(\beta+1)}$. There are more terms in $t\hat{c}_t$ but its leading terms are just opposite to the leading terms in $\hat{c}$. $\square$

Thus we can choose the lower bound of $\hat{c}$ in condition $(v)$ to be

$$
C^-(t,x) = \begin{cases} -k_1 t^{-(\beta+1)}(1+|x|)^{-\alpha}, & \text{if } 1+|x| \leq kt^{\frac{\beta+1}{2-\alpha}}, \\ 0, & \text{if } 1+|x| > kt^{\frac{\beta+1}{2-\alpha}}, \end{cases}
$$

where $k = (\frac{k_1}{k_2})^{\frac{1}{2-\alpha}}$.

To bound the integral in $(v)$, we use the estimates

$$
\begin{aligned}
w^{-1}(T,x)P(T,x) \;=\;& w^{-1}\frac{3}{4}\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1}\frac{w}{T'} \\
=\;& \frac{3}{4}T\left(6 + \frac{S(x)}{T}\right)^{-1}\frac{1}{T'} \\
\leq\;& C\frac{T}{T'} \\
\leq\;& C\frac{t^{\beta+1}}{t^{\beta}} \\
=\;& Ct.
\end{aligned}
$$

and

$$\begin{aligned}
w^{-1}(T,x) &= T^m e^{\frac{S(x)}{T}} \\
&\leq T^m e^{\frac{(1+|x|)^{2-\alpha}}{T}} \\
&\leq t^{m(\beta+1)} e^{\frac{(1+|x|)^{2-\alpha}}{t^{\beta+1}}}.
\end{aligned}$$

On the support of $C^-$, we have

$$\begin{aligned}
w^{-1}(T,x) &\leq t^{m(\beta+1)} e^{t^{\frac{\beta+1}{2-\alpha}(2-\alpha)} t^{-(\beta+1)}} \\
&= t^{m(\beta+1)}.
\end{aligned}$$

The condition to check becomes

$$\begin{aligned}
\int_{\mathbf{R}^n} P w^{-2} |C^-|^{\frac{p+1}{p-1}}\, dx &\leq C t^{1+m(\beta+1)-(\beta+1)\frac{p+1}{p-1}} \int_{|x|\leq k t^{\frac{\beta+1}{2-\alpha}}} (1+|x|)^{-\alpha\frac{p+1}{p-1}}\, dx, \\
&= C t^{1+(\beta+1)(m-\frac{p+1}{p-1})} \int_{|x|\leq k t^{\frac{\beta+1}{2-\alpha}}} (1+|x|)^{-\alpha\frac{p+1}{p-1}}\, dx.
\end{aligned}$$

To estimate the integral on the right, we use Theorem (1.2.1)

$$\begin{aligned}
\int_{|x|\leq k t^{\frac{\beta+1}{2-\alpha}}} (1+|x|)^{-\alpha\frac{p+1}{p-1}}\, dx &= \int_0^{t^{\frac{\beta+1}{2-\alpha}}} \left( \int_{\partial B(0,s)} (1+s)^{-\alpha\frac{p+1}{p-1}}\, d\sigma \right) ds \\
&= \int_0^{t^{\frac{\beta+1}{2-\alpha}}} (1+s)^{-\alpha\frac{p+1}{p-1}}\ mes\ \partial B(0,s)\, ds \\
&\leq \int_0^{t^{\frac{\beta+1}{2-\alpha}}} s^{-\alpha\frac{p+1}{p-1}}\ n\sigma(n) s^{n-1}\, ds \\
&= C t^{\frac{\beta+1}{2-\alpha}\left(n-\alpha\frac{p+1}{p-1}\right)}.
\end{aligned}$$

Hence for every $\delta > 0$ we have

$$\int_{|x| \le kt^{\frac{\beta+1}{2-\alpha}}} (1+|x|)^{-\alpha\frac{p+1}{p-1}}\, dx \le \begin{cases} Ct^{-\frac{\alpha(\beta+1)}{2-\alpha}\frac{p+1}{p-1}+\frac{n(\beta+1)}{2-\alpha}+\delta} & \text{, if } \alpha\frac{p+1}{p-1} \ge n, \\ C & \text{, if } \alpha\frac{p+1}{p-1} < n. \end{cases}$$

Thus

$$\int_{\mathbf{R}^n} Pw^{-2}|C^-|^{\frac{p+1}{p-1}}\, dx \le \begin{cases} Ct^{1+(\beta+1)(m-\frac{p+1}{p-1})+\frac{\beta+1}{2-\alpha}(-\alpha\frac{p+1}{p-1}+n)+\delta} & \text{, if } \alpha\frac{p+1}{p-1} \ge n, \\ Ct^{1+(\beta+1)(m-\frac{p+1}{p-1})} & \text{, if } \alpha\frac{p+1}{p-1} < n. \end{cases}$$

Clearly the right sides are bounded functions of $t \ge t_0$ if $m$ is defined as (3.3.5).

The condition for integrability $(i)$ leads to similar estimates; we can show that

$$\int_{t_0}^{\infty}\int_{\mathbf{R}^n} w^{-1}|C^-|^{\frac{p+1}{p-1}}\, dxdt < \infty,$$

for the same $m$ and $\gamma$ defined in (3.3.5). Finally we can verify the inequality in condition $(i)$ using

$$t^{-1}P \le Cw, \quad |P_t| \le Cw,$$

together with

$$\hat{c} \ge C^-, \quad -t\hat{c}_t \le C^-$$

from the auxiliary Lemma (3.3.8). The resulting lower bound is

$$\begin{aligned} \hat{c}w - (\hat{c}P)_t &= \hat{c}w - t\hat{c}_t \cdot t^{-1}P - \hat{c}P_t \\ &\ge CC^-w. \end{aligned}$$

Thus conditions $(i)$ and $(v)$ hold if Lemma (3.3.8) holds.

### 3.3.2 Proofs of Main Theorems and Corollaries

Now we are ready to proof the decay estimates for the solution of the nonlinear wave equation (3.0.1).

*Proof.* **(Proof of Theorem (1.3.4))** We choose weights $P$ and $w$ defined in (3.3.2) with parameters $m = m(\lambda) - 2\delta$ and $\gamma = m(\lambda) - \delta$, see (3.3.4). Then Proposition (3.3.3) yields the following weighted estimates:

$$\begin{aligned}
\int_{\mathbf{R}^n} a(t,x) w^{-1} u^2 \, dx &\leq CE_0, \\
\int_{\mathbf{R}^n} \lambda(x)\eta(t) w^{-1} u^2 \, dx &\leq CE_0, \\
\int_{\mathbf{R}^n} \lambda(x) w^{-1} u^2 \, dx &\leq CE_0 T', \\
\int_{\mathbf{R}^n} \lambda(x) T^m e^{\frac{S(x)}{T}} u^2 \, dx &\leq CE_0 T', \\
\int_{\mathbf{R}^n} \lambda(x) e^{\frac{S(x)}{T}} u^2 \, dx &\leq CE_0 T' T^{-m}, \\
\int_{\mathbf{R}^n} \lambda(x) e^{(m(\lambda)-\delta)\frac{\phi(x)}{T}} u^2 \, dx &\leq CE_0 T^{2\delta - m(\lambda)} T', \\
\int_{\mathbf{R}^n} \lambda(x) e^{(m(\lambda)-\delta)\frac{\phi(x)}{t^{\beta+1}}} u^2 \, dx &\leq CE_0 t^{(\beta+1)(2\delta - m(\lambda))} t^{\beta},
\end{aligned}$$

where $t \geq t_0$ and $C$ is a constant depending on $E_0$. To simplify these we recall

$$\phi_0 (1+|x|)^{2-\alpha} \leq \phi(x) \leq \phi_1 (1+|x|)^{2-\alpha}.$$

Thus $\lambda(x) \geq C(\phi(x))^{-\frac{\alpha}{2-\alpha}}$ with some $C > 0$. We have the lower bounds

$$\lambda(x) \geq CT^{-\frac{\alpha}{2-\alpha}} \left( \frac{\phi(x)}{T} \right)^{-\frac{\alpha}{2-\alpha}} \geq CT^{-\frac{\alpha}{2-\alpha}} e^{-\delta \frac{\phi(x)}{T}} \geq Ct^{-(\beta+1)\frac{\alpha}{2-\alpha}} e^{-\delta \frac{\phi(x)}{t^{\beta+1}}}.$$

$$
\begin{aligned}
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} u^2 &\leq CE_0 t^{\beta+(\beta+1)(2\delta-m(\lambda))+(\beta+1)(\frac{\alpha}{2-\alpha})}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} u^2 &\leq CE_0 t^{\beta+(\beta+1)(2\delta-m(\lambda)+\frac{\alpha}{2-\alpha})}.
\end{aligned}
$$

For the second estimate we have

$$
\begin{aligned}
\int_{\mathbf{R}^n} Pw^{-2}(u_t^2 + |\nabla u|^2)\ dx &\leq CE_0, \\
\int_{\mathbf{R}^n} \frac{3}{4}\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1} \frac{T^m}{T'} e^{\frac{S(x)}{T}} (u_t^2 + |\nabla u|^2) dx &\leq CE_0, \\
\int_{\mathbf{R}^n} \left(\frac{1}{T} + \frac{\phi(x)}{T^2}\right)^{-1} e^{(m(\lambda)-\delta)\frac{\phi(x)}{T}} (u_t^2 + |\nabla u|^2) dx &\leq CE_0 T' T^{2\delta - m(\lambda)}.
\end{aligned}
$$

Now

$$
\left(\frac{1}{T} + \frac{\phi(x)}{T^2}\right)^{-1} = T\left(1 + \frac{\phi(x)}{T}\right)^{-1} \geq Te^{-\delta\frac{\phi(x)}{T}},
$$

$$
\begin{aligned}
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} (u_t^2 + |\nabla u|^2) dx &\leq CE_0 t^{\beta} t^{(\beta+1)(2\delta-m(\lambda))} t^{-(\beta+1)}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} (u_t^2 + |\nabla u|^2) dx &\leq CE_0 t^{(\beta+1)(2\delta-m(\lambda))-1}.
\end{aligned}
$$

The last estimate is

$$
\int_{\mathbf{R}^n} Pw^{-2}|u|^{p+1}\ dx \leq CE_0.
$$

As in the previous estimate we have

$$
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} |u|^{p+1} dx \leq CE_0 t^{(\beta+1)(2\delta-m(\lambda))-1}.
$$

$\square$

*Proof.* **(Proof of Theorem (1.3.5))** Here we use the weights $P$ and $w$ with the second set of parameters defined in (3.3.5). The result is again a simple consequence of Proposition (3.3.3) and the lower bounds on $\lambda(x)$ and $\left(\frac{1}{T}+\frac{\phi(x)}{T^2}\right)^{-1}$. $\square$

*Proof.* **(Proof of Corollary (3.1.3))** We add the three estimates in Theorem (1.3.4), or Theorem (1.3.5), and restrict the integration to $\{x : \phi(x) \geq T^{1+\epsilon}\}$ :

$$\int_{\phi(x)\geq t^{(1+\beta)(1+\epsilon)}} e^{(m(\lambda)-\delta)\frac{\phi(x)}{t^{1+\beta}}}(u^2+u_t^2+|\nabla u|^2+|u|^{p+1})\,dx \;\;\leq\;\; CE_0t^c,$$

where $C$ depends on $\alpha$, $p$, and $n$. From $\phi(x)/t^{\beta+1} \geq t^{\epsilon(1+\beta)}$ we have that

$$\int_{\phi(x)\geq t^{(1+\beta)(1+\epsilon)}} (u^2+u_t^2+|\nabla u|^2+|u|^{p+1})\,dx \;\;\leq\;\; CE_0t^ce^{-(m(\lambda)-\delta)t^{\epsilon(1+\beta)}}.$$

This completes the proof, since $t^c$ can be included in the exponential term after a slight increase of $\delta$. $\square$

## 3.4 Case two: $\lambda(x)$ and $\eta(t)$ are given by (3.0.5) and (3.0.6)

**Proposition 3.4.1.** *Let $P$ and $w$ be positive weights satisfying conditions (i)–(iv) in Proposition (3.2.1) and the following condition:*

$$(v) \qquad (1-\epsilon)P\Big(w_t+(1-\epsilon)aw\Big) > w^2,$$

*where $\epsilon \in (0,1)$. Assume that $a(t,x)=\lambda(x)\eta(t)$, where $\lambda(x)$ and $\eta(t)$ are defined by (2.0.3) and (2.0.4) respectively. Assume that condition (2.0.5) also satisfied. Then the following* *estimate holds for the solution $u$ of problem (2.0.1):*

$$\begin{aligned}\int_{\mathbf{R}^n} P(v_t^2 + |\nabla v|^2)\, dx &\leq CE_0,\\ \int_{\mathbf{R}^n} awv^2\, dx &\leq CE_0,\\ \int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\, dx &\leq CE_0.\end{aligned}$$

*for $t \geq t_0$.*

*Proof.* We begin with the result of Proposition (3.2.1):

$$\begin{aligned}2E_0 \geq & \int_{\mathbf{R}^n} \Big(P(v_t^2 + |\nabla v|^2) + 2wv_tv + (\hat{c}P + w_t + aw)v^2\Big)\, dx\\ & + \frac{2}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\, dx.\end{aligned}$$

This equation can be rewritten as

$$\begin{aligned}2E_0 \geq & \int_{\mathbf{R}^n} \Big((1-\epsilon)P(v_t^2 + |\nabla v|^2) + 2wv_tv + (\hat{c}P + w_t + (1-\epsilon)aw)v^2\Big)dx\\ & + \int_{\mathbf{R}^n} \epsilon P(v_t^2 + |\nabla v|^2)dx + \int_{\mathbf{R}^n} \epsilon awv^2 dx\\ & + \frac{2}{p+1}\int_{\mathbf{R}^n} Pw^{p-1}|v|^{p+1}\, dx,\end{aligned}$$

for $\epsilon \in (0, 1)$. Condition $(v)$ insure that the following quadratic form

$$(1-\epsilon)Pv_t^2 + 2wv_tv + \big(w_t + (1-\epsilon)aw\big)v^2 \geq 0,$$

is positive definite. □

**Proposition 3.4.2.** *Assume that $a(t, x) = \lambda(x)\eta(t)$, where $\lambda(x)$ and $\eta(t)$ are defined by (3.0.5), (3.0.6), such that condition (2.0.5) satisfied. Also $P$ and $w$ satisfy $(i)-(v)$ and the* *following condition also satisfy*

$$(vi) \qquad Pw^{-3}(w_t^2 + |\nabla w|^2) \leq Ca(t,x),$$

*where $C$ is a constant. Then the solution $u$ of (3.0.1)–(3.0.2) satisfies*

$$\begin{aligned}
\int_{\mathbf{R}^n} aw^{-1}u^2 \, dx &\leq 2(E_0+1), \\
\int_{\mathbf{R}^n} Pw^{-2}(u_t^2 + |\nabla u|^2) \, dx &\leq 2(E_0+1), \\
\int_{\mathbf{R}^n} Pw^{-2}|u|^{p+1} \, dx &\leq 2(E_0+1),
\end{aligned}$$

*for $t \geq t_0$.*

The above result yields non-trivial estimates if there exist weights with properties $(i)-(vi)$ which decay sufficiently fast as $t$ and $|x|$ go to infinity. Such weights $P$ and $w$ are already constructed for the linear equation in [20]. Here we find a new pair of weights adapted to the nonlinear term in problem (3.0.1). Both weights are defined in terms of certain positive solutions to the Poisson equation in $\mathbf{R}^n$. We provide more details in the next section.

### 3.4.1 Construction of Weights

In this section we define a family of weights $P$ and $w$ satisfying conditions $(i)-(vi)$. We rely on the results from [44], [45] and Proposition (2.1.1). Given the parameter $S_0 > 0$ and $\gamma > 0$, we introduce

$$S(x) = \gamma\phi(x) + S_0, \quad x \in \mathbf{R^n},$$

where $\phi(x)$ is the positive solution of the Poisson equation (2.1.5). We are using the same weights as in the previous chapter:

$$w(T,x) = T^{-m}e^{-\frac{S(x)}{T}}, \quad P(T,x) = \frac{3}{4}\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1}\frac{w(T,x)}{T'}. \tag{3.4.1}$$

Note that $S$, $P$, $w > 0$ on $\mathbf{R}_+ \times \mathbf{R}^n$. $T$ is defined above (3.1.3), and $m > 0$ is an additional parameter. The two numbers $\gamma$ and $m$ depend on $n$, $p$ and $a$. We need to show the additional conditions that have been introduced due to the existence of absorption terms; see Proposition (2.3.2) for the proof of conditions $(ii)-(vi)$. The final condition $(i)$ is related to the asymptotic behavior of $w$ as a solution to

$$w_{TT} - \Delta w + \lambda(x)w_T \geq C^- w. \tag{3.4.2}$$

We choose $C^-$ so that $m$ is maximized. The choice of $C^-$ is different if the exponent $p$ of the nonlinear term is large or small. The parameters $m$ and $\gamma$ have to be chosen such that condition $(i)$ is satisfied.

*The choice of the parameters $m$ and $\gamma$ for the* ***supercritical*** *exponents $p$.*

**Proposition 3.4.3.** *Let $P$ and $w$ be defined in (3.4.1), while $m$ and $\gamma$ be defined as follows:*

$$m = m(\lambda) - 2\delta, \quad \gamma = m(\lambda) - \delta. \tag{3.4.3}$$

*Then condition $(i)$ holds for sufficiently large $t \geq t_0$.*

*Proof.* First we need $\hat{c} \geq 0$ and $\hat{c}_t \leq 0$ for large $t \geq t_0$, see Proposition (2.3.2) for the proofs. As a consequence of $\hat{c} \geq 0$ and $\hat{c}_t \leq 0$ for large $t \geq t_0$ we can take $C^- = 0$. Thus the integrals in conditions $(i)$ are bounded. Moreover, from these assumptions and condition $(ii)$ we find

$$\hat{c}w - (\hat{c}P)_t = \hat{c}(w - P_t) - \hat{c}_t P \geq 0.$$

Thus the inequality in condition $(i)$ holds. $\square$

*The choice of the parameters $m$ and $\gamma$ for the* ***subcritical*** *exponents $p$*

**Proposition 3.4.4.** *Let $P$ and $w$ be defined in (3.4.1), while $m$ and $\gamma$ are defined as follows:*

$$
\begin{aligned}
m &= -\frac{1}{1-\beta} + \frac{p+1}{p-1} - \frac{\alpha}{2+\alpha}\frac{p+1}{p-1} - \frac{n}{2+\alpha} - \delta, \\
\gamma &= m(\lambda) - \delta,
\end{aligned}
\tag{3.4.4}
$$

*where $\delta > 0$ is a small number. Then condition $(i)$ holds for sufficiently large $t \geq t_0$.*

Before proving Proposition (3.4.4) we need to find a lower bound for $\hat{c}$ and an upper bound for $\hat{c}_t$.

**Lemma 3.4.5.** *There exist positive constants $k_i$, $i = 1, 2, 3, 4$, such that*

$$
\begin{aligned}
\hat{c}(t,x) &\geq -k_1(1+|x|)^{\alpha}(1+t)^{\beta-1} + k_2(1+|x|)^{2+2\alpha}(1+t)^{2(\beta-1)}, \\
(1+t)\hat{c}_t(t,x) &\leq k_3(1+|x|)^{\alpha}(1+t)^{\beta-1} - k_4(1+|x|)^{2+2\alpha}(1+t)^{2(\beta-1)}.
\end{aligned}
$$

*Proof.* Using inequality (A.3) from Lemma (A.3), we obtain

$$
\begin{aligned}
\hat{c} &\geq \left(-\frac{m}{T} + \frac{S(x)}{T^2}\right)T'' + \left(\frac{m}{T^2} - \frac{2S(x)}{T^3}\right)T'^2 + \frac{\lambda(x)S(x) - |\nabla S(x)|^2}{T^2} + \\
&+ \frac{\Delta S(x) - m\lambda(x)}{T}.
\end{aligned}
$$

Recall that $S(x) = \gamma\phi(x) + S_0$, where $\gamma$ is defined by (3.4.4) and $\phi(x)$ is the positive solution of the Poisson equation. The main difference between this proposition and the previous one is that $m$ can be larger than $m(\lambda)$.

Using $(S1)$ of Lemma (2.3.1),

$$\begin{aligned}\Delta S(x) - m\lambda(x) &= (m(\lambda)-\delta)\triangle\phi(x) - m\lambda(x)\\ &= (m(\lambda)-\delta)\lambda(x) - m\lambda(x)\\ &= (m(\lambda)-m-\delta)\lambda(x)\\ &\geq -k_1(1+|x|)^\alpha.\end{aligned}$$

However, by condition $(S3)$ of Lemma (2.3.1),

$$\begin{aligned}\lambda(x)S(x) - |\nabla S(x)|^2 &\geq \lambda(x)S(x) - \left(1-\frac{\delta}{2m(\lambda)}\right)\lambda(x)S(x)\\ &= \frac{\delta}{2m(\lambda)}\lambda(x)S(x)\\ &\geq k_2(1+|x|)^\alpha(1+|x|)^{2+\alpha}\\ &= k_2(1+|x|)^{2+2\alpha},\ \text{where } k_2>0.\end{aligned}$$

Hence

$$\begin{aligned}\hat{c} &\geq -k_1(1+t)^{\beta-1}(1+|x|)^\alpha + k_2(1+t)^{2\beta-2}(1+|x|)^{2+2\alpha} - \frac{m(1+t)^{-\beta-1}}{t^{1-\beta}}\\ &+ \frac{S(x)t^{-\beta-1}}{(1+t)^{2-2\beta}} + \frac{m(1+t)^{-2\beta}}{(1+t)^{2-2\beta}} - \frac{2S(x)t^{2\beta}}{(1+t)^{3\beta+3}}\\ &= -k_1(1+t)^{\beta-1}(1+|x|)^\alpha + k_2(1+t)^{2\beta-2}(1+|x|)^{2+2\alpha} - S(x)(1+t)^{\beta-3}\\ &\geq -k_1(1+t)^{\beta-1}(1+|x|)^\alpha + k_2(1+t)^{2\beta-2}(1+|x|)^{2+2\alpha},\end{aligned}$$

for sufficiently large $t \geq t_0$. The estimate of $(1+t)\hat{c}_t$ is very similar, since $\hat{c}$ is a polynomial of $(1+t)^{\beta-1}$. There are more terms in $(1+t)\hat{c}_t$ but its leading terms are just opposite to the corresponding terms in $\hat{c}$. $\square$

Hence we can choose the lower bound of $\hat{c}$ in condition $(i)$ to be

$$C^-(t,x) = \begin{cases} -k_1(1+t)^{\beta-1}(1+|x|)^\alpha, & \text{if } \ 1+|x| \leq k(1+t)^{\frac{1-\beta}{2+\alpha}}, \\ 0, & \text{if } \ 1+|x| > k(1+t)^{\frac{1-\beta}{2+\alpha}}, \end{cases}$$

where $k = (\frac{k_1}{k_2})^{\frac{1}{2+\alpha}}$.

To verify the integral condition in $(i)$, we use the estimates

$$\begin{aligned} w^{-1}(T,x) &= T^m e^{\frac{S(x)}{T}} \\ &\leq T^m e^{\frac{(1+|x|)^{2+\alpha}}{T}} \\ &\leq (1+t)^{m(1-\beta)} e^{\frac{(1+|x|)^{2+\alpha}}{(1+t)^{1-\beta}}}, \end{aligned}$$

so on the support of $C^-$

$$\begin{aligned} w^{-1}(T,x) &\leq C(1+t)^{m(1-\beta)} e^{(1+t)^{\frac{1-\beta}{2+\alpha}(2+\alpha)}(1+t)^{-(1-\beta)}} \\ &= C(1+t)^{m(1-\beta)}. \end{aligned}$$

Using the above estimate together with $C^-$,

$$\begin{aligned} \int_{t_0}^\infty \int_{\mathbf{R}^n} w^{-1}|C^-|^{\frac{p+1}{p-1}}\, dxdt &\leq C\int_{t_0}^\infty (1+t)^{m(1-\beta)}(1+t)^{(\beta-1)\frac{p+1}{p-1}} \int_{\mathbf{R}^n} (1+|x|)^{\alpha\frac{p+1}{p-1}}\, dxdt, \\ &\leq C\int_{t_0}^\infty (1+t)^{(m-\frac{p+1}{p-1})(1-\beta)} \int_{|x|\leq(1+t)^{\frac{1-\beta}{2+\alpha}}} (1+|x|)^{\alpha\frac{p+1}{p-1}}\, dxdt. \end{aligned}$$

To estimate the second integral on the right we use Theorem (1.2.1):

$$\begin{aligned}
\int_{|x|\le(1+t)^{\frac{1-\beta}{2+\alpha}}}(1+|x|)^{\alpha\frac{p+1}{p-1}}\,dx &= \int_0^{(1+t)^{\frac{1-\beta}{2+\alpha}}}\left(\int_{\partial B(0,s)}(1+s)^{\alpha\frac{p+1}{p-1}}\,d\sigma\right)ds,\\
&= \int_0^{(1+t)^{\frac{1-\beta}{2+\alpha}}}(1+s)^{\alpha\frac{p+1}{p-1}}\,\mathrm{meas}\left(\partial B(0,s)\right)\,ds,\\
&\le \int_0^{(1+t)^{\frac{1-\beta}{2+\alpha}}}(1+s)^{\alpha\frac{p+1}{p-1}}\,n\sigma(n)s^{n-1}\,ds,\\
&= C(1+t)^{\frac{1-\beta}{2+\alpha}\left(n+\alpha\frac{p+1}{p-1}\right)}.
\end{aligned}$$

For every $\delta>0$ we have

$$\int_{|x|\le k(1+t)^{\frac{1-\beta}{2+\alpha}}}(1+|x|)^{\alpha\frac{p+1}{p-1}}\,dx \le \begin{cases} C(1+t)^{\frac{1-\beta}{2+\alpha}\left(n+\alpha\frac{p+1}{p-1}\right)+\delta}, & \text{if } -\alpha\frac{p+1}{p-1}\ge n,\\ C, & \text{if } -\alpha\frac{p+1}{p-1}<n.\end{cases}$$

Thus

$$\int_{t_0}^{\infty}\int_{\mathbf{R}^n} w^{-1}|C^-|^{\frac{p+1}{p-1}}\,dxdt \le \begin{cases} C\int_{t_0}^{\infty}(1+t)^{(1-\beta)(m-\frac{p+1}{p-1}+\frac{n}{2+\alpha}+\frac{\alpha}{2+\alpha}\frac{p+1}{p-1})+\delta}, & \text{if } -\alpha\frac{p+1}{p-1}\ge n,\\ C\int_{t_0}^{\infty}(1+t)^{(1-\beta)(m-\frac{p+1}{p-1})}, & \text{if } -\alpha\frac{p+1}{p-1}<n,\end{cases}$$

$$\int_{t_0}^{\infty}\int_{\mathbf{R}^n} w^{-1}|C^-|^{\frac{p+1}{p-1}}\,dxdt \le \begin{cases} C(1+t)^{1+(1-\beta)(m-\frac{p+1}{p-1}+\frac{n}{2+\alpha}+\frac{\alpha}{2+\alpha}\frac{p+1}{p-1})+\delta}, & \text{if } -\alpha\frac{p+1}{p-1}\ge n,\\ C(1+t)^{1+(1-\beta)(m-\frac{p+1}{p-1})}, & \text{if } -\alpha\frac{p+1}{p-1}<n.\end{cases}$$

Clearly the right sides are bounded functions of $t\ge t_0$ if $m$ is defined as (3.4.4). Finally we can verify the inequality in condition $(i)$ using

$$t^{-1}P\le Cw,\quad |P_t|\le Cw,$$

together with

$$\hat{c}\ge C^-,\quad -t\hat{c}_t\le C^-$$

from Lemma (3.4.5). The resulting lower bound is

$$\begin{aligned}\hat{c}w - (\hat{c}P)_t &= \hat{c}w - t\hat{c}_t \cdot t^{-1}P - \hat{c}P_t \\ &\geq CC^{-}w.\end{aligned}$$

Thus condition $(i)$ holds if Lemma (3.4.5) holds.

### 3.4.2 Proofs of Main Theorems and Corollaries

Now we are ready to prove the decay estimates for the solution of the nonlinear wave equation (3.0.1).

*Proof.* **(Proof of Theorem (1.3.12))** Let the weights $P$ and $w$ be defined in (3.4.1) with parameters $m = m(\lambda) - 2\delta$ and $\gamma = m(\lambda) - \delta$, see (3.4.3). Then Proposition (3.4.2) yields the following weighted estimates:

$$\begin{aligned}\int_{\mathbf{R}^n} a(t,x)w^{-1}u^2\,dx &\leq CE_0, \\ \int_{\mathbf{R}^n} \lambda(x)\eta(t)w^{-1}u^2\,dx &\leq CE_0, \\ \int_{\mathbf{R}^n} \lambda(x)w^{-1}u^2\,dx &\leq CE_0T', \\ \int_{\mathbf{R}^n} \lambda(x)T^m e^{\frac{S(x)}{T}}u^2\,dx &\leq CE_0T', \\ \int_{\mathbf{R}^n} \lambda(x)e^{\frac{S(x)}{T}}u^2\,dx &\leq CE_0T'T^{-m}, \\ \int_{\mathbf{R}^n} \lambda(x)e^{(m(\lambda)-\delta)\frac{\phi(x)}{T}}u^2\,dx &\leq CE_0T^{2\delta-m(\lambda)}T', \\ \int_{\mathbf{R}^n} \lambda(x)e^{(m(\lambda)-\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}}u^2\,dx &\leq CE_0(1+t)^{-\beta-(m(\lambda)-2\delta)(1-\beta)}.\end{aligned}$$

Since $\lambda(x) \geq \lambda_0$, we obtain

$$\int_{\mathbf{R}^n} e^{(m(\lambda)-\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}} u^2 \, dx \quad \leq \quad CE_0(1+t)^{-\beta-(m(\lambda)-2\delta)(1-\beta)}.$$

For the second estimate we have

$$\begin{aligned}
\int_{\mathbf{R}^n} Pw^{-2}(u_t^2 + |\nabla u|^2) \, dx \quad &\leq \quad CE_0, \\
\int_{\mathbf{R}^n} \frac{3}{4}\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1} \frac{T^m}{T'} e^{\frac{S(x)}{T}} (u_t^2 + |\nabla u|^2) dx \quad &\leq \quad CE_0, \\
\int_{\mathbf{R}^n} \left(\frac{1}{T} + \frac{\phi(x)}{T^2}\right)^{-1} e^{(m(\lambda)-\delta)\frac{\phi(x)}{T}} (u_t^2 + |\nabla u|^2) dx \quad &\leq \quad CE_0 T' T^{2\delta - m(\lambda)}.
\end{aligned}$$

Now

$$\left(\frac{1}{T} + \frac{\phi(x)}{T^2}\right)^{-1} = T\left(1 + \frac{\phi(x)}{T}\right)^{-1} \geq Te^{-\delta\frac{\phi(x)}{T}},$$

$$\begin{aligned}
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{T}} (u_t^2 + |\nabla u|^2) dx \quad &\leq \quad CE_0 T^{-1} T' T^{2\delta - m(\lambda)}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{1-\beta}}} (u_t^2 + |\nabla u|^2) dx \quad &\leq \quad CE_0(1+t)^{-\beta}(1+t)^{(1-\beta)(2\delta - m(\lambda))}(1+t)^{-(1-\beta)}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} (u_t^2 + |\nabla u|^2) dx \quad &\leq \quad CE_0(1+t)^{-\beta+(1-\beta)(2\delta - m(\lambda)-1)}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} (u_t^2 + |\nabla u|^2) dx \quad &\leq \quad CE_0(1+t)^{(1-\beta)(2\delta - m(\lambda))-1}.
\end{aligned}$$

The last estimate is

$$\int_{\mathbf{R}^n} Pw^{-2}|u|^{p+1} \, dx \quad \leq \quad CE_0.$$

As in the previous estimate we have

$$\int_{\mathbf{R}^n} e^{(m(\lambda)-2\delta)\frac{\phi(x)}{(1+t)^{\beta+1}}}|u|^{p+1}dx \;\; \leq \;\; CE_0 t^{(1-\beta)(2\delta-m(\lambda))-1}.$$

$\square$

*Proof.* **(Proof of Theorem (1.3.13))** Here we use the weights $P$ and $w$ with the second set of parameters defined in (3.4.4). The result is again a simple consequence of Proposition (3.4.2) and the lower bounds on $\lambda(x)$ and $\left(\frac{1}{T}+\frac{\phi(x)}{T^2}\right)^{-1}$. $\square$

*Proof.* **(Proof of Corollary (1.3.14))** We add the three estimates in Theorem (1.3.12), or Theorem (1.3.13), and restrict the integration to $\{x : \phi(x) \geq (1+t)^{(1-\beta)(1+\epsilon)}\}$:

$$\int_{\phi(x)\geq(1+t)^{(1-\beta)(1+\epsilon)}} e^{(m(\lambda)-\delta)\frac{\phi(x)}{(1+t)^{1-\beta}}}\left(u^2+u_t^2+|\nabla u|^2+|u|^{p+1}\right) dx \;\; \leq \;\; CE_0(1+t)^c,$$

where $C$ depends on $\alpha$, $p$, and $n$. From $\phi(x)/(1+t)^{1-\beta} \geq t^{\epsilon(1-\beta)}$ we have that

$$\int_{\phi(x)\geq(1+t)^{(1-\beta)(1+\epsilon)}} \left(u^2+u_t^2+|\nabla u|^2+|u|^{p+1}\right) dx \;\; \leq \;\; CE_0(1+t)^c e^{-(m(\lambda)-\delta)t^{\epsilon(1-\beta)}}.$$

This completes the proof, since $(1+t)^c$ can be included in the exponential term after a slight increase of $\delta$. $\square$

# Chapter 4

# The Critical Exponent

In this chapter we are going to estimate the critical exponent for the damped wave equation with focusing nonlinearity

$$u_{tt} - \Delta u + a(t,x)u_t = |u|^p, \quad x \in \mathbf{R}^n, \quad t > 0, \tag{4.0.1}$$

$$u(0,x) = \varepsilon u_0(x), \quad u_t(0,x) = \varepsilon u_1(x), \tag{4.0.2}$$

where $\varepsilon > 0$ and $(u_0, u_1)$ are compactly supported data from the energy space

$$u_0 \in H^1(\mathbf{R}^n), \quad u_1 \in L^2(\mathbf{R}^n),$$
$$\operatorname{supp} u_i \subset B(R) := \{x \in \mathbf{R}^n : |x| < R\}, \quad i = 0, 1.$$

We consider exponents $p$ which satisfy $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$ and $1 < p < \infty$ for $n = 1, 2$. The potential $a(t,x) \in C^1(\mathbf{R}_+, \mathbf{R^n})$ is a positive function, such that

$$a(t,x) \sim a_0(1+|x|)^{-\alpha}(1+t)^{-\beta}, \quad |x| + t \to \infty,$$

with $a_0 > 0$, $\alpha \in [0,1)$, $\beta \in (-1,1)$ and $0 < \alpha + \beta < 1$. We can show that problem (4.0.1)–(4.0.2) has a unique global solution for sufficiently small $\varepsilon$ if $p > p_c(n,\alpha,\beta)$, where the critical exponent $p_c(n,\alpha,\beta)$ is defined in (1.3.10).

## 4.1 Global Existence of the Solution

The local existence and uniqueness is a simple modification of the result in Strauss [40]: problem (4.0.1)–(4.0.2) admits a unique local solution $u \in X_1(0,T_m)$, where $X_1(0,T_m)$ is defined by (1.1.1), for some positive $T_m = T_m(\epsilon)$. If the lifespan $T_m < \infty$, then

$$\limsup_{t\to T_m} (\|u_t(t,.)\|_{L^2}^2 + \|\nabla u(t,.)\|_{L^2}^2) = \infty. \tag{4.1.1}$$

Hence $u$ can be extended to a global solution if its energy norm does not blow up in finite time. This fact is known as the continuation principle for evolutionary equations. The finite speed of propagation also holds:

$$\operatorname{supp} u(t,.) \subset B(R+t), \quad t \in [0,T_m).$$

We are interested in the critical exponent, namely to find a critical number $p_{cr}(n)$, such that one of the following holds:

- if $1 < p \le p_{cr}(n)$ some solutions of (4.0.1)–(4.0.2) blow-up in finite time, regardless of the smallness and smoothness of the initial data;
- if $p_{cr}(n) < p < \frac{n+2}{n-2}$ all small data solutions of (4.0.1)–(4.0.2) are global.

To show the global existence of the solution $u \in [0, T_m(\epsilon)) \times \mathbf{R}^\mathbf{n}$ for small initial data, we rely on the work of Todorova and Yordanov [44].

First we find an approximate solution $w$ of (4.0.1). Rewrite (4.0.1) for separable damping terms $a(t,x) = \lambda(x)\eta(t)$:

$$u_{tt} - \Delta u + \lambda(x)\eta(t)u_t = |u|^{p-1}u,$$

where $\lambda(x), \eta(t)$ are positive $C^1$ coefficients as defined in (3.0.3), (3.0.4). Due to the diffusion phenomenon, the approximate solution of (4.0.1) is also an approximate solution of the corresponding parabolic equation

$$\lambda(x)\eta(t)w_t - \Delta w = 0. \tag{4.1.2}$$

Using the same transformation as in Chapter 1, we introduce a new parameter

$$T(t) = \int_0^t \frac{ds}{\eta(s)},$$

such that $T \to \infty$ as $t \to \infty$; in fact, $T \sim (1+t)^{1-\beta}$, $T' = \frac{1}{\eta(t)} \sim (1+t)^{-\beta}$ as $t \to \infty$. Since $w$ will be an approximate solution of

$$\lambda(x)w_T - \Delta w = 0, \tag{4.1.3}$$

$w$ will be given by

$$w(T,x) = T^{-m}e^{-\gamma\frac{\phi(x)}{T}}, \tag{4.1.4}$$

with suitably chosen parameters $m$, $\gamma$. Here $\phi(x) \in C^2(\mathbf{R}^n)$ is the positive solution of the Poisson equation

$$\Delta\phi(x) = \lambda(x), \quad x \in \mathbf{R}^n; \tag{4.1.5}$$

recall that $\lambda(x)$ is given by (3.0.3) while $\phi(x)$ has properties $(a1)-(a3)$ as given in Chapter 2 and Proposition (3.1.1).

Next we derive a modified equation for $v = w^{-1}u$ and weighted energy identities.

It is easy to check that the transformed equation on $[0, T_m(\epsilon))$ is

$$v_{tt} - \triangle v + \hat{a} v_t + \hat{b} \cdot \nabla v + \hat{c} v = w^{p-1}|v|^p, \tag{4.1.6}$$

where the coefficients are given by

$$\begin{aligned} \hat{a} &= a(t,x) + 2w_t w^{-1}, \\ \hat{b} &= -2w^{-1} \cdot \nabla w, \\ \hat{c} &= w^{-1}(w_{tt} - \triangle w + a(t,x) w_t). \end{aligned} \tag{4.1.7}$$

To find the weighted energy multiply the transformed equation (4.1.6) by $Pv_t + wv$ with weights $P, w \in C^2((0,\infty) \times \mathbf{R}^n)$. Then we integrate over $\mathbf{R^n}$ to derive the following energy identity:

$$\frac{d}{dt} E(v_t, \nabla v, v) + F(v_t, \nabla v) + G(v) = H(v) + \frac{1}{p+1} \frac{d}{dt} \int_{\mathbf{R^n}} P w^{p-1} |v|^p \, dx, \tag{4.1.8}$$

for $t \in [0, T_m(\epsilon))$ where

$$E(v_t, \nabla v, v) = \frac{1}{2} \int_{\mathbf{R^n}} \left( P(v_t^2 + |\nabla v|^2) + 2wv_t v + (\hat{c}P + w_t + aw) v^2 \right) dx, \tag{4.1.9}$$

$$\begin{aligned} F(v_t, \nabla v) &= \frac{1}{2} \int_{\mathbf{R^n}} (-P_t + 2aP + 4P(\ln w)_t - 2w) v_t^2 \, dx \\ &+ \int_{\mathbf{R^n}} (\nabla P - 2P \nabla \ln w) \cdot v_t \nabla v \; dx \\ &+ \frac{1}{2} \int_{\mathbf{R^n}} (-P_t + 2w) |\nabla v|^2 \; dx, \end{aligned} \tag{4.1.10}$$

$$G(v) = \int_{\mathbf{R^n}} (\hat{c}w - (\hat{c}P)_t) v^2 \; dx, \tag{4.1.11}$$

$$H(v) = \int_{\mathbf{R^n}} \left( w^p - \frac{1}{p+1} (w^{p-1} P)_t \right) |v|^{p+1} \; dx. \tag{4.1.12}$$

See the proof in Appendix $B$ with $k = -1$.

We need different conditions on the damping and the weights $P$ and $w$ to insure that $F(v_t, \nabla v) + G(v) > 0$ and hence the weighted energy $E(v_t, \nabla v, v)$ is bounded.

**Lemma 4.1.1.** *Suppose that $a(t,x) = \lambda(x)\eta(t)$, with $\lambda(x)$, and $\eta(t)$ are defined by (3.0.3), (3.0.4). There exists a large number $t_0 > 0$ such that for $t \geq t_0$, the following conditions hold:*

$$
\begin{aligned}
(i) \qquad & \hat{c} \geq 0, \quad \hat{c}_t \leq 0, \quad \hat{c}w - (\hat{c}P)_t \geq 0. \\
(ii) \qquad & -P_t + w \geq 0, \\
(iii) \qquad & (-P_t + 2w)(-P_t + 2aP + 4P(\ln w)_t - 2w) \geq (\nabla P - 2P\nabla \ln w)^2.
\end{aligned}
$$

*If $u$ is a solution of (4.0.1) for $t \in (t_0, T_m)$, we have*

$$
\begin{aligned}
E(v_t, \nabla v, v)(t) \leq E(v_t, \nabla v, v)\mid_{t=0} \quad &+ \quad \underbrace{\frac{1}{p+1}\int_{\mathbf{R^n}} P w^{p-1} |v|^{p+1} \, dx}_{(I)} \\
&+ \quad \underbrace{\int_{t_0}^{t} H(s)\, ds}_{(II)} . \qquad (4.1.13)
\end{aligned}
$$

*Proof.* See Section (3.3) for the proof of $(i)-(iii)$. Since $F(v_t, \nabla v)$ is given by (4.1.10), it is a positive definite quadratic form. Therefore, condition $(iii)$ implies $F(v_t, \nabla v) \geq 0$. Also using condition $(i)$ we have $G(v) \geq 0$. Therefore, integrating (4.1.8) over $[t_0, t]$, where $t_0 < t < T_m$ we have inequality (4.1.13). □

We need to estimate the space dependent norm $(I)$ and the space–time dependent norm $(II)$ of equation (4.1.13). An important estimate will be used to control these norms. With the weight $1 + \frac{S(x)}{t^{\beta+1}}$ we are gaining some decay on the right hand side.

**Lemma 4.1.2.** *Let $a(t,x) = \lambda(x)\eta(t)$, where $\lambda(x)$, $\eta(t)$ are defined by (3.0.3), (3.0.4). If $p > p_c(n, \alpha, \beta)$, then there is a number $\xi > 0$, which depends on $p$, $n$, $\alpha$, $\beta$ and $\delta$ such that*

$$\int_{\mathbf{R^n}} \left(1 + \frac{S(x)}{t^{\beta+1}}\right) P w^{p-1} |v|^{p+1} \, dx \le C t^{-\xi} W(t)^{\frac{p+1}{2}},$$

*where*

$$\begin{aligned} \xi &= (\beta+1)\Bigg((mp-1) - (p+1)\left(\frac{\theta}{2}(m + \frac{\alpha}{2-\alpha}) + \frac{1-\theta}{2}(m-1)\right)\Bigg) \\ &- \beta\left(\frac{p+1}{2} - 1\right) \\ &= (\beta+1)\left(\frac{(p-1)(n-\alpha)-2}{2-\alpha} - \delta(p-1)\right) - \beta\left(\frac{p-1}{2}\right) \end{aligned}$$

*and*

$$W(t) = \int_{\mathbf{R^n}} \lambda(x)\eta(t) w(T,x) v^2 \, dx + \int_{\mathbf{R^n}} P(T,x)(v_t^2 + |\nabla v|^2) \, dx.$$

*Proof.* Using the definitions of $P$ and $w$ we have

$$
\begin{aligned}
\frac{S(x)}{T}Pw^{p-1}|v|^{p+1} &= \frac{3}{4}\frac{S(x)}{T}\left(\frac{6}{T}+\frac{S(x)}{T^2}\right)^{-1}\frac{w}{T'}w^{p-1}|v|^{p+1}\\
&= \frac{3}{4}\frac{S(x)}{T}T\left(6+\frac{S(x)}{T}\right)^{-1}\frac{w^p}{T'}|v|^{p+1}\\
&\leq \frac{3}{4}\frac{S(x)}{T}T\frac{T}{S(x)}\frac{w^p}{T'}|v|^{p+1}\\
&= C\frac{T}{T'}w^p|v|^{p+1},\\
Pw^{p-1}|v|^{p+1} &= \frac{3}{4}\left(\frac{6}{T}+\frac{S(x)}{T^2}\right)^{-1}\frac{w}{T'}w^{p-1}|v|^{p+1}\\
&= \frac{3}{4}T\left(6+\frac{S(x)}{T}\right)^{-1}\frac{w^p}{T'}|v|^{p+1}\\
&\leq \frac{3}{4}\frac{1}{6}T\frac{w^p}{T'}|v|^{p+1}\\
&\leq C\frac{T}{T'}w^p|v|^{p+1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{\mathbf{R^n}}\left(1+\frac{S(x)}{T}\right)Pw^{p-1}|v|^{p+1}\,dx &\leq C\frac{T}{T'}\int_{\mathbf{R^n}}w^p|v|^{p+1}\,dx\\
&= C\frac{T}{T'}\int_{\mathbf{R^n}}T^{-mp}e^{-p\frac{S(x)}{T}}|v|^{p+1}\,dx\\
&\leq C\frac{T^{1-mp}}{T'}\int_{\mathbf{R^n}}e^{-p\frac{S(x)}{T}}|v|^{p+1}\,dx. \qquad (4.1.14)
\end{aligned}
$$

Define

$$\psi(T,x)=\frac{1}{2}\frac{S(x)}{T},\quad \mu=\frac{2p}{p+1}.$$

Equation (4.1.14) can be written as

$$
\begin{aligned}
\int_{\mathbf{R^n}}\left(1+\frac{S(x)}{T}\right)Pw^{p-1}|v|^{p+1}\,dx &\leq C\frac{T^{1-mp}}{T'}\int_{\mathbf{R^n}}e^{-\mu\psi(T,x)(p+1)}|v|^{p+1}\,dx\\
&= C\frac{T^{1-mp}}{T'}\|e^{-\mu\psi(T,x)}v\|_{p+1}^{p+1}. \qquad (4.1.15)
\end{aligned}
$$

We continue with the Gagliardo–Nirenberg estimate:

$$\|e^{-\mu\psi}v\|_{p+1} \leq \underbrace{\|e^{-\mu\psi}v\|^{\theta}}_{A}\underbrace{\|\nabla(e^{-\mu\psi}v)\|^{1-\theta}}_{B}, \tag{4.1.16}$$

where

$$\theta = 1 - n\left(\frac{1}{2} - \frac{1}{p+1}\right).$$

To estimating term $A$ of (4.1.16), we have

$$\begin{aligned} e^{-2\mu\psi}v^2 &= e^{-\frac{S(x)}{T}}T^{-m}T^{m}e^{-\frac{p-1}{p+1}\frac{S(x)}{T}}T^{-\frac{\alpha}{2-\alpha}}T^{\frac{\alpha}{2-\alpha}}v^2 \\ &= T^{m}w(T,x)e^{-\frac{p-1}{p+1}\frac{S(x)}{T}}T^{-\frac{\alpha}{2-\alpha}}T^{\frac{\alpha}{2-\alpha}}v^2. \end{aligned} \tag{4.1.17}$$

There exists a constant $C$ such that for any $y > 0$ it is true that

$$y^{\frac{\alpha}{2-\alpha}} \leq Ce^{y^{\frac{p-1}{p+1}}}$$

whenever $\alpha \in [0,1)$ and $\frac{\alpha}{2-\alpha} < 1$. Thus,

$$y^{-\frac{\alpha}{2-\alpha}} \geq Ce^{-y^{\frac{p-1}{p+1}}}$$

and

$$\begin{aligned} e^{-2\mu\psi}v^2 &\leq T^{m}w(T,x)\left(\frac{S(x)}{T}\right)^{-\frac{\alpha}{2-\alpha}}T^{-\frac{\alpha}{2-\alpha}}T^{\frac{\alpha}{2-\alpha}}v^2 \\ &= T^{m+\frac{\alpha}{2-\alpha}}w(T,x)v^2\bigg(m(\lambda) - 2\delta\bigg)^{-\frac{\alpha}{2-\alpha}}\phi(x)^{-\frac{\alpha}{2-\alpha}} \\ &\leq CT^{m+\frac{\alpha}{2-\alpha}}w(T,x)v^2\lambda(x)\frac{\eta(t)}{\eta(t)}. \end{aligned}$$

The above inequality implies

$$\|e^{-\mu\psi}v\|^2 \le CT^{m+\frac{\alpha}{2-\alpha}}T' \int_{\mathbf{R^n}} \lambda(x)\eta(t)w(T,x)v^2\ dx. \tag{4.1.18}$$

To estimate term $(B)$ of (4.1.16), we begin with

$$|\nabla(e^{-\mu\psi}v)|^2 = \underbrace{\mu^2 e^{-2\mu\psi}|\nabla\psi|^2 v^2}_{B1} - \underbrace{2\mu e^{-2\mu\psi}v\nabla\psi\cdot\nabla v}_{B2} + \underbrace{e^{-2\mu\psi}|\nabla v|^2}_{B3}$$

and integrate by parts in $(B2)$:

$$\int_{\mathbf{R^n}} 2\mu e^{-2\mu\psi}v\nabla\psi\cdot\nabla v\ dx = 2\mu^2\int_{\mathbf{R^n}} e^{-2\mu\psi}v^2|\nabla\psi|^2\ dx - \mu\int_{\mathbf{R^n}} e^{-2\mu\psi}v^2\triangle\psi\ dx.$$

Combining (B1), (B2), and (B3), we have

$$\begin{aligned}
\int_{\mathbf{R^n}} |\nabla(e^{-\mu\psi}v)|^2\ dx &= \int_{\mathbf{R^n}} \mu^2 e^{-2\mu\psi}|\nabla\psi|^2v^2 - \int_{\mathbf{R^n}} 2\mu e^{-2\mu\psi}v\nabla\psi\cdot\nabla v + \int_{\mathbf{R^n}} e^{-2\mu\psi}|\nabla v|^2 \\
&= \mu^2\int_{\mathbf{R^n}} e^{-2\mu\psi}|\nabla\psi|^2v^2 - 2\mu^2\int_{\mathbf{R^n}} e^{-2\mu\psi}v^2|\nabla\psi|^2\ dx \\
&+ \mu\int_{\mathbf{R^n}} e^{-2\mu\psi}v^2\triangle\psi\ dx + \int_{\mathbf{R^n}} e^{-2\mu\psi}|\nabla v|^2\ dx \\
&= -\mu^2\int_{\mathbf{R^n}} e^{-2\mu\psi}|\nabla\psi|^2v^2 + \mu\int_{\mathbf{R^n}} e^{-2\mu\psi}v^2\triangle\psi\ dx \\
&+ \int_{\mathbf{R^n}} e^{-2\mu\psi}|\nabla v|^2\ dx \\
&\le \underbrace{\mu\int_{\mathbf{R^n}} e^{-2\mu\psi}v^2\triangle\psi\ dx}_{B4} + \underbrace{\int_{\mathbf{R^n}} e^{-2\mu\psi}|\nabla v|^2\ dx}_{B5}.
\end{aligned} \tag{4.1.19}$$

To estimate term $(B4)$ of (4.1.19) we use

$$\begin{aligned} \triangle\psi(T,x) &= \frac{1}{2T}\triangle S(x) \\ &= \frac{m(\lambda)-\delta}{2T}\triangle\phi(x) \\ &= \frac{m(\lambda)-\delta}{2T}\lambda(x)\frac{\eta(t)}{\eta(t)}. \end{aligned}$$

Hence we obtain

$$\int_{\mathbf{R^n}} e^{-2\mu\psi}v^2\triangle\psi\ dx = \frac{(m(\lambda)-\delta)T'}{2T}\int_{\mathbf{R^n}}\lambda(x)\eta(t)e^{-2\mu\psi}v^2\ dx.$$

Finally, the exponential term satisfies

$$\begin{aligned} e^{-2\mu\psi} &= e^{\frac{-2p}{p+1}\frac{S(x)}{T}} \\ &= e^{-\frac{S(x)}{T}}T^{-m}T^m e^{-\frac{p-1}{p+1}\frac{S(x)}{T}} \\ &\leq w(T,x)T^m, \end{aligned}$$

so $(B4)$ is estimated as follows:

$$\mu\int_{\mathbf{R^n}} e^{-2\mu\psi}v^2\triangle\psi\ dx \leq CT^{m-1}T'\int_{\mathbf{R^n}}\lambda(x)\eta(t)w(T,x)v^2\ dx. \tag{4.1.20}$$

To proceed with the estimate of term $(B5)$ in (4.1.19), we need

$$\begin{aligned} e^{-2\mu\psi} &= e^{\frac{-2p}{p+1}\frac{S(x)}{T}} \\ &= e^{-\frac{S(x)}{T}}T^{-m}T^m e^{-\frac{p-1}{p+1}\frac{S(x)}{T}} \\ &= \frac{4}{3}\left(\frac{6}{T}+\frac{S(x)}{T^2}\right)T'P(T,x)T^m e^{-\frac{p-1}{p+1}\frac{S(x)}{T}} \\ &= \frac{4}{3}\left(6+\frac{S(x)}{T}\right)\frac{T'}{T}P(T,x)T^m e^{-\frac{p-1}{p+1}\frac{S(x)}{T}}. \end{aligned}$$

For all $y \geq 0$, $(6+y)e^{-ky} \leq C$. Thus we have

$$e^{-2\mu\psi} \leq CT'T^{m-1}P(T,x)$$

and estimate $(B5)$ as follows:

$$\int_{\mathbf{R^n}} e^{-2\mu\psi}|\nabla v|^2 \, dx \leq CT'T^{m-1}\int_{\mathbf{R^n}} P(T,x)|\nabla v|^2 \, dx. \tag{4.1.21}$$

Therefore, using estimates (4.1.18), (4.1.20) and (4.1.21), we can rewrite inequality (4.1.16) as

$$\begin{aligned}\|e^{-\mu\psi}v\|_{p+1}^{p+1} \leq \; & C\left(T^{m+\frac{\alpha}{2-\alpha}}T'\int_{\mathbf{R^n}} \lambda(x)\eta(t)w(T,x)v^2dx\right)^{\frac{\theta}{2}(p+1)} \\ & \left(T^{m-1}T'\left(\int_{\mathbf{R^n}} \lambda(x)\eta(t)w(T,x)v^2dx + \int_{\mathbf{R^n}} P(T,x)|\nabla v|^2dx\right)\right)^{\frac{1-\theta}{2}(p+1)}.\end{aligned}$$

Let us now define

$$W(t) = \int_{\mathbf{R^n}} \lambda(x)\eta(t)w(T,x)v^2 \, dx + \int_{\mathbf{R^n}} P(T,x)(v_t^2 + |\nabla v|^2) \, dx. \tag{4.1.22}$$

Then

$$\begin{aligned}\|e^{-\mu\psi}v\|_{p+1}^{p+1} & \leq CT^{\frac{\theta}{2}(m+\frac{\alpha}{2-\alpha})(p+1)}T'^{\frac{\theta}{2}(p+1)}W(t)^{\frac{\theta}{2}(p+1)}T^{\frac{(1-\theta)}{2}(m-1)(p+1)}W(t)^{\frac{1-\theta}{2}(p+1)}T'^{\frac{p+1}{2}} \\ & \leq CT^{\frac{\theta(p+1)}{2}(m+\frac{\alpha}{2-\alpha})+\frac{(1-\theta)(p+1)}{2}(m-1)}T'^{\frac{p+1}{2}}W(t)^{\frac{p+1}{2}}\end{aligned}$$

and

$$\int_{\mathbf{R^n}}\left(1+\frac{S(x)}{T}\right)Pw^{p-1}|v|^{p+1}dx \leq CT^{1-mp}T^{\frac{\theta(p+1)}{2}(m+\frac{\alpha}{2-\alpha})+\frac{(1-\theta)(p+1)}{2}(m-1)}T'^{\frac{p+1}{2}-1}W(t)^{\frac{p+1}{2}}.$$

It is convenient to introduce

$$
\begin{aligned}
\xi &= (\beta+1)\Bigg((mp-1)-(p+1)\Big(\frac{\theta}{2}(m+\frac{\alpha}{2-\alpha})+\frac{1-\theta}{2}(m-1)\Big)\Bigg)-\beta\Big(\frac{p+1}{2}-1\Big)\\
&= (\beta+1)\Big(\frac{(p-1)(n-\alpha)-2}{2-\alpha}-\delta(p-1)\Big)-\beta\Big(\frac{p-1}{2}\Big).
\end{aligned}
$$

Our assumption $p > p_c(n,\alpha,\beta)$ means that $\xi > 0$ :

$$
\begin{aligned}
0 &< (\beta+1)\Big(\frac{(p-1)(n-\alpha)-2}{2-\alpha}-\delta(p-1)\Big)-\beta\Big(\frac{p-1}{2}\Big),\\
0 &< \frac{(\beta+1)(p-1)(n-\alpha)}{2-\alpha}-\frac{2(\beta+1)}{2-\alpha}-\delta(p-1)(\beta+1)-\beta\frac{p-1}{2},\\
\frac{2(\beta+1)}{2-\alpha} &< (p-1)\frac{2(\beta+1)(n-\alpha)-2\delta(\beta+1)(2-\alpha)-\beta(2-\alpha)}{2(2-\alpha)},\\
p-1 &> \frac{4(\beta+1)}{2(\beta+1)(n-\alpha)-2\delta(\beta+1)(2-\alpha)-\beta(2-\alpha)},\\
p &> \frac{4(\beta+1)}{2(\beta+1)(n-\alpha)-\beta(2-\alpha)}+1.
\end{aligned}
$$

□

Using the results of Lemma (4.1.2), we are going to estimate the term $(II)$ of inequality (4.1.13) in Lemma (4.1.1).

**Lemma 4.1.3.** *Under the assumptions of Lemma (4.1.2) we have*

$$
\int_{t_0}^{t} H(s)\ ds \ \leq\ C\int_{t_0}^{t} s^{-1-\xi}W(s)^{\frac{p+1}{2}}\ ds, \quad t\in[t,T_m),
$$

*where $C>0$, $\xi>0$ are defined in Lemma (4.1.2).*

*Proof.* From the definition of $P$ and $w$, we have

$$\begin{aligned} K(T,x) &= w^p - \frac{1}{p+1}(Pw^{p-1})_t \\ &= w^p - \frac{1}{p+1}\Big(P_t w^{p-1} + (p-1)Pw^{p-2}w_t\Big) \\ &= Pw^{p-1}\Big(\frac{w}{P} - \frac{1}{p+1}\frac{P_t}{P} - \frac{p-1}{p+1}\frac{w_t}{w}\Big) \\ &= Pw^{p-1}\Big(\frac{w}{P} - \frac{1}{p+1}\frac{P_t}{P} - \frac{p}{p+1}\frac{w_t}{w} + \frac{1}{p+1}\frac{w_t}{w}\Big). \end{aligned}$$

It is easy to see that

$$\frac{w_t}{w} = \Big(-\frac{m}{T} + \frac{S(x)}{T^2}\Big)T'$$

and

$$\frac{w}{P} = \frac{4}{3}\Big(\frac{6}{T} + \frac{S(x)}{T^2}\Big)T'.$$

Moreover,

$$P_t = -\frac{3}{4}\Big(\frac{6}{T} + \frac{S(x)}{T^2}\Big)^{-2}\Big(-\frac{6T'}{T^2} - \frac{2S(x)TT'}{T^4}\Big)\frac{w}{T'} + \frac{3}{4}\Big(\frac{6}{T} + \frac{S(x)}{T^2}\Big)^{-1}\frac{w_tT' - wT''}{T'^2}$$

and

$$\frac{P_t}{P} = \Big(\frac{6}{T} + \frac{S(x)}{T^2}\Big)^{-1}\Big(\frac{6T'}{T^2} + \frac{2S(x)T'}{T^3}\Big) + \frac{w_t}{w} - \frac{T''}{T'}.$$

Hence, we obtain

$$
\begin{aligned}
K(T,x) &= Pw^{p-1}\Big[\frac{4}{3}\Big(\frac{6}{T}+\frac{S(x)}{T^2}\Big)T' - \frac{T'}{p+1}\Big(\frac{6}{T}+\frac{S(x)}{T^2}\Big)^{-1}\Big(\frac{6}{T^2}+\frac{2S(x)}{T^3}\Big) \\
&- \frac{1}{p+1}\frac{w_t}{w}+\frac{1}{p+1}\frac{T''}{T'}-\frac{p}{p+1}\Big(-\frac{m}{T}+\frac{S(x)}{T^2}\Big)T'+\frac{1}{p+1}\frac{w_t}{w}\Big] \\
&\leq Pw^{p-1}\Big[\frac{4}{3}\Big(\frac{6}{T}+\frac{S(x)}{T^2}\Big)T'+\frac{1}{p+1}\frac{T''}{T'}+\frac{mp}{p+1}\frac{T'}{T}\Big] \\
&\leq CPw^{p-1}\Big[\Big(8+\frac{4}{3}\frac{S(x)}{T}\Big)\frac{T'}{T}+\frac{T''}{T'}+\frac{T'}{T}\Big] \\
&= CPw^{p-1}\Big[\Big(9+\frac{4}{3}\frac{S(x)}{T}\Big)\frac{T'}{T}+\frac{T''}{T'}\Big] \\
&\leq CPw^{p-1}\Big[\Big(1+\frac{S(x)}{T}\Big)\frac{T'}{T}+\frac{T''}{T'}\Big].
\end{aligned}
$$

Since

$$\frac{T'}{T}\ ,\ \frac{T''}{T'} \sim t^{-1},$$

the estimate of $K(T,x)$ simplifies to

$$K(T(t),x) \leq Ct^{-1}Pw^{p-1}\Big(1+\frac{S(x)}{t^{\beta+1}}\Big).$$

If we integrate over $[t_0,t]$,

$$\int_{t_0}^{t} H(s)\ ds \ \ \leq \ \ \int_{t_0}^{t} s^{-1}\int_{\mathbf{R^n}}\Big(1+\frac{S(x)}{s^{\beta+1}}\Big)Pw^{p-1}|v|^{p+1}\ dsdx.$$

Now we just apply Lemma (4.1.2) to derive the estimate in Lemma (4.1.3). $\square$

Using Lemma (4.1.2), (4.1.3) and the weighted energy identity (4.1.13), we have

$$
\begin{aligned}
E(v_t,\nabla v)(t) &\leq E(v_t,\nabla v)(t_0)+Ct^{-\xi}W(t)^{\frac{p+1}{2}}+\int_{t_0}^{t} s^{-1-\xi}W(s)^{\frac{p+1}{2}}ds, \\
&\leq E(v_t,\nabla v)(t_0)+CW(t)^{\frac{p+1}{2}}+C\int_{t_0}^{t} s^{-1-\xi}W(s)^{\frac{p+1}{2}}ds. \qquad (4.1.23)
\end{aligned}
$$

Inequality (4.1.23) can be written as

$$\begin{aligned} E(v_t, \nabla v, v) &= \frac{1}{2}\int_{\mathbf{R^n}} \Big(P(v_t^2 + |\nabla v|^2) + 2wv_t v + (\hat{c}P + w_t + aw)v^2\Big)dx \\ &\le E(v_t, \nabla v)(t_0) + CW(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\xi}W(s)^{\frac{p+1}{2}}ds. \end{aligned} \tag{4.1.24}$$

We know that both $P, \hat{c} \ge 0$ and $2wv_tv + w_tv^2 = \frac{d}{dt}(wv^2)$. Thus,

$$\begin{aligned} &\frac{1}{2}\int_{\mathbf{R^n}} P(v_t^2 + |\nabla v|^2)dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbf{R^n}} wv^2 dx + \frac{1}{2}\int_{\mathbf{R^n}} a(t,x)wv^2 dx \\ &\qquad \le E(v_t, \nabla v)(t_0) + CW(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\xi}W(s)^{\frac{p+1}{2}}ds. \end{aligned} \tag{4.1.25}$$

**Lemma 4.1.4.** *Let $\alpha \in [0,1)$, $\beta \in (-1,1)$, $d > 0$, $d_0 > 0$, $R > 0$ and $E_0 > 0$ be given real numbers. Assume that $f \in C([t_0, T_m))$ is a monotone increasing function. If a function $h \in C^1([t_0, T_m))$ satisfies*

$$h'(t) + \frac{d_0}{(d+t+R)^{\alpha+\beta}}h(t) \le E_0 + f(t),$$

*then*

$$h(t) \le h(t_0) + C(E_0 + f(t))(d+t+R)^{\alpha+\beta}, \quad t \ge t_0, \textit{where the constant } C > 0.$$

*Proof.* From the identity

$$\frac{d}{dt}\Big(e^{\frac{d_0}{1-\alpha-\beta}(d+t+R)^{1-\alpha-\beta}}h(t)\Big) = e^{\frac{d_0}{1-\alpha-\beta}(d+t+R)^{1-\alpha-\beta}}\Big(h'(t) + \frac{d_0}{(d+t+R)^{\alpha+\beta}}h(t)\Big),$$

we obtain

$$\frac{d}{dt}\Big(e^{\frac{d_0}{1-\alpha-\beta}(d+t+R)^{1-\alpha-\beta}}h(t)\Big) \le e^{\frac{d_0}{1-\alpha-\beta}(d+t+R)^{1-\alpha-\beta}}\Big(E_0 + f(t)\Big).$$

Now integrating over $[t_0, t]$, where $t < T_m$ we have

$$
\begin{aligned}
e^{\frac{d_0}{1-\alpha-\beta}(d+t+R)^{1-\alpha-\beta}} h(t) &\leq e^{\frac{d_0}{1-\alpha-\beta}(d+t_0+R)^{1-\alpha-\beta}} h(t_0) + E_0 \int_{t_0}^{t} e^{\frac{d_0}{1-\alpha-\beta}(d+s+R)^{1-\alpha-\beta}} ds \\
&+ \int_{t_0}^{t} e^{\frac{d_0}{1-\alpha-\beta}(d+s+R)^{1-\alpha-\beta}} f(s) ds \\
&\leq e^{\frac{d_0}{1-\alpha-\beta}(d+t+R)^{1-\alpha-\beta}} h(t_0) \\
&+ \Big(E_0 + f(t)\Big) \int_{t_0}^{t} e^{\frac{d_0}{1-\alpha-\beta}(d+s+R)^{1-\alpha-\beta}} ds,
\end{aligned}
$$

using the fact that $f(t)$ is a monotone increasing function. Since

$$
\int_{t_0}^{t} e^{\frac{d_0}{1-\alpha-\beta}(d+s+R)^{1-\alpha-\beta}} ds = \int_{d+t_0+R}^{d+t+R} e^{C\tau^{1-\alpha-\beta}} d\tau
$$

$$
= \frac{1}{1-\alpha-\beta} \int_{(d+t_0+R)^{1-\alpha-\beta}}^{(d+t+R)^{1-\alpha-\beta}} e^{Cz} z^{\frac{\alpha}{1-\alpha-\beta}} dz \leq e^{C(d+t+R)^{1-\alpha-\beta}} \frac{(d+t+R)^{\alpha+\beta}}{C(1-\alpha-\beta)},
$$

where $C = d_0/(1-\alpha-\beta)$, we have the desired estimate. $\square$

Define

$$
M(t) := \max_{0 \leq s \leq t} W(s). \tag{4.1.26}
$$

Note that the function $t \mapsto M(t)$ is monotone increasing. After these preparations we can prove

**Lemma 4.1.5.** *Let $\alpha \in [0,1)$, $\beta \in (-1,1)$, with $0 < \alpha+\beta < 1$. Then*

$$
\int_{\mathbf{R^n}} wv^2 \; dx \leq \int_{\mathbf{R^n}} wv^2 \; dx\big|_{t=t_0} + Ct^{\alpha+\beta}\Big(E(v_t, \nabla v)(t_0) + W(t)^{\frac{p+1}{2}} + \int_{t_0}^{t} s^{-1-\xi} W(t)^{\frac{p+1}{2}} \; ds\Big)
$$

*for $t \in [t_0, T_m)$ with large $t_0 > 0$.*

*Proof.* It follows from (4.1.25) that

$$\begin{aligned}\frac{1}{2}\int_{\mathbf{R}^n} a(t,x)wv^2dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^n} wv^2dx \;&\le\; E(v_t,\nabla v)(t_0) + CW(t)^{(p+1)/2}\\ &+\; C\int_{t_0}^{t} s^{-1-\gamma}W(s)^{(p+1)/2}ds, \quad t\in[t_0,T_m).\end{aligned}$$

Using $a(t,x)\sim a_0(1+|x|)^{-\alpha}(1+t)^{-\beta}$ and (4.1.26), we get

$$\begin{aligned}\frac{1}{2}\frac{a_0}{(1+t+R)^{\alpha+\beta}}\int_{\mathbf{R}^n} wv^2dx + \frac{d}{dt}\int_{\mathbf{R}^n} wv^2dx \;&\le\; 2E(v_t,\nabla v)(t_0) + CM(t)^{\frac{p+1}{2}}\\ &+\; C\int_{t_0}^{t} s^{-1-\gamma}W(s)^{(p+1)/2}ds, \quad t\in[t_0,T_m).\end{aligned}$$

Since the function

$$t\mapsto CM(t)^{\frac{p+1}{2}} + C\int_{t_0}^{t} s^{-1-\gamma}W(s)^{(p+1)/2}ds$$

is monotone increasing, we can apply Lemma (4.1.5) with

$$h(t)=\int_{\mathbf{R}^n} wv^2dx, \quad E_0 = 2E(v_t,\nabla v)(t_0), \quad a=1, \quad d_0=a_0,$$

$$f(t) = CM(t)^{\frac{p+1}{2}} + \int_{t_0}^{t} s^{-1-\gamma}W(s)^{(p+1)/2}ds);$$

this yields the desired estimate. $\square$

Denote $E_u(t)=\frac{1}{2}(\|u_t(t,\cdot)\|^2+\|\nabla u(t,\cdot)\|^2)$, where $u\in X_1(0,T_m)$ is the weak solution of problem (4.0.1)–(4.0.2). We need the following result.

**Lemma 4.1.6.** *For each $t\in[0,T_m)$ it is true that*

$$\int_{\mathbf{R}^n} w(T,x)v^2(T,x)\ dx \le C_R(t)\|\nabla u(T,\cdot)\|^2,$$

$$E(v_t,\nabla v)(t)\le C_R(t)E_u(t),$$

*with some $t$-dependent constant $C_R(t)$ satisfying* $\lim_{t\to+\infty} C_R(t) = \infty$.

*Proof.* Recall that $v = \frac{u}{w}$ and $w(T,x) = T^{-m}e^{\frac{-\phi(x)}{T}}$. Thus we have

$$\begin{aligned}\int_{\mathbf{R}^n} w(T,x)v^2(T,x)\,dx &= \int_{\mathbf{R}^n} \frac{u^2}{w}\,dx,\\ &= \int_{\mathbf{R}^n} u^2 T^m e^{\frac{\phi(x)}{T}}\,dx.\end{aligned}$$

Using $\phi_0(1+|x|)^{2-\alpha} \le \phi(x) \le \phi_1(1+|x|)^{2-\alpha}$ and the finite speed of propagation,

$$\begin{aligned}\int_{\mathbf{R}^n} w(T,x)v^2(T,x)\,dx &\le \int_{\mathbf{R}^n} u^2(T,x)t^{m(\beta+1)}e^{\frac{\phi_1(1+t+R)^{2-\alpha}}{t^{\beta+1}}}\,dx,\\ &\le C_R(t)\int_{\mathbf{R}^n} u^2(T,x)\,dx.\end{aligned}$$

Now the claim follows from the Poincaré inequality. $\square$

The standard energy identity associated with problem (4.0.1)–(4.0.2) is

$$E_u(t) \le E_u(0) + \frac{1}{p+1}\|u(t,\cdot)\|_{p+1}^{p+1}. \tag{4.1.27}$$

**Lemma 4.1.7.** *Let $t_0 > 0$ be the time defined in Lemmas (4.1.1)–(4.1.6). Then there exists $T^* \in (t_0, T_m)$, which depends on $\varepsilon > 0$, such that for all $t \in [0, T^*]$*

$$E_u(t) \le 2E_u(0) = (\|u_1\|^2 + \|\nabla u_0\|^2)\varepsilon^2,$$

$$\lim_{\varepsilon\to 0} T^*(\varepsilon) = \infty.$$

*Proof.* Using the Gagliardo-Nirenberg inequality, Poincaré inequality, and the finite speed of propagation in (4.1.27), we get

$$\begin{aligned} E_u(t) &\leq E_u(0) + \frac{1}{p+1}\|\nabla u(t,\cdot)\|^{(1-\theta)(p+1)}\|u(t,\cdot)\|^{\theta(p+1)} \\ &\leq E_u(0) + C(1+t+R)^{\theta(p+1)}\|\nabla u(t,\cdot)\|^{(1-\theta)(p+1)}\|\nabla u(t,\cdot)\|^{\theta(p+1)} \\ &\leq E_u(0) + C(1+t+R)^{\theta(p+1)}\|\nabla u(t,\cdot)\|^{p+1}. \end{aligned}$$

Thus we have

$$E_u(t) \leq E_u(0) + C(t+R)^{\theta(p+1)}E_u(t)^{\frac{p+1}{2}}, \tag{4.1.28}$$

where $\theta = 1 - n\big(\frac{1}{2} - \frac{1}{p+1}\big)$, and $C > 0$. Denote by $T^*$ the first time $T^* > 0$, such that $E_u(T^*) = 2E_u(0)$. Since $E_u(0) < 2E_u(0)$, it follows that $E_u(t) < 2E_u(0)$ for all $t \in [0, T^*)$. From (4.1.28) with $t = T^*$, we have

$$2E_u(0) \leq E_u(0) + C(T+R)^{\theta(p+1)}(2E_u(0))^{\frac{p+1}{2}}.$$

By solving this inequality for $T^*$, find the following lower bound for $T^*$:

$$T^* \geq CE_u(0)^{\frac{1-p}{2\theta(p+1)}} - R$$

$$= C(\varepsilon^2)^{\frac{1-p}{2\theta(p+1)}} - R.$$

This implies that by taking $\varepsilon > 0$ sufficiently small we can make a desired relation $t_0 < T^* < T_m$, where $T_m$ is the life span of the solution. Note that the $T^*$ depends on $\varepsilon$, hence $T^*(\varepsilon) \to \infty$ when $\varepsilon \to 0$. $\square$

## 4.2 Proofs of Main Results

In this section we are going to prove the main theorems and the corollaries, but first we need to state several propositions in order to find decay estimates for the energy, $L^2$ and $L^{p+1}$ norms.

**Proposition 4.2.1.** *Assume that $a(t,x) = \lambda(x)\eta(t)$, where $\lambda(x)$, $\eta(t)$ are defined by (3.0.3), (3.0.4). If the weights $P$ and $w$ satisfy conditions (i)–(iii) and*

$$
\begin{aligned}
(iv) \qquad & w \le C_1 t^{-(\alpha+\beta)} P, \\
(v) \qquad & |w_t| \le C_1 t^{-(\alpha+\beta)} w,
\end{aligned}
$$

*we have that*

$$
\begin{aligned}
\int_{\mathbf{R^n}} P(v_t^2 + |\nabla v|^2)\ dx &\le C\varepsilon^2, \\
\int_{\mathbf{R^n}} awv^2\ dx &\le C\varepsilon^2,
\end{aligned}
$$

*for all $t \ge t_0$.*

*Proof.* Lemma (4.1.7) shows that we can consider only the case $t_0 < T^* < T_m$. Using inequality (4.1.24),

$$
\begin{aligned}
\int_{\mathbf{R^n}} \left( P(v_t^2 + |\nabla v|^2) + 2wv_tv + (\hat{c}P + w_t + aw)v^2 \right) dx \;&\le\; 2E(v_t, \nabla v)(t_0) + CW(t)^{\frac{p+1}{2}} \\
&+\; C\int_{t_0}^{t} s^{-1-\xi} W(s)^{\frac{p+1}{2}}\ ds.
\end{aligned}
$$

Since $\hat{c}P > 0$, we have

$$\begin{aligned}\int_{\mathbf{R^n}} \Big( P(v_t^2 + |\nabla v|^2) + 2wv_tv + (w_t + aw)v^2 \Big) dx &\leq 2E(v_t, \nabla v)(t_0) + CW(t)^{\frac{p+1}{2}} \\ &+ C\int_{t_0}^{t} s^{-1-\xi} W(s)^{\frac{p+1}{2}} ds. \qquad (4.2.1)\end{aligned}$$

The following inequality holds for every $\epsilon \in (0,1)$:

$$|2wv_tv| \leq \epsilon t^{\alpha+\beta} wv_t^2 + \epsilon^{-1} t^{-(\alpha+\beta)} wv^2.$$

Hence estimate (4.2.1) implies

$$\begin{aligned}&\int_{\mathbf{R}^n} P(v_t^2 + |\nabla v|^2)\ dx - \epsilon t^{\alpha+\beta} \int_{\mathbf{R}^n} wv_t^2\ dx \\ &-\epsilon^{-1} t^{-(\alpha+\beta)} \int_{\mathbf{R}^n} wv^2\ dx + \int_{\mathbf{R}^n} (w_t + aw)v^2\ dx \\ \leq\ & 2E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C\int_{t_0}^{t} s^{-1-\gamma} W(s)^{(p+1)/2}\ ds.\end{aligned}$$

Combining like terms,

$$\begin{aligned}&\int_{\mathbf{R}^n} (P - \epsilon t^{\alpha+\beta} w)(v_t^2 + |\nabla v|^2)\ dx + \int_{\mathbf{R}^n} (w_t + aw - \epsilon^{-1} t^{-(\alpha+\beta)} w)v^2\ dx \\ \leq\ & 2E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C\int_{t_0}^{t} s^{-1-\gamma} W(s)^{(p+1)/2}\ ds, \quad t \in [t_0, T_m).\end{aligned}$$

Here conditions $(iv)$ and $(v)$ yield

$$\begin{aligned}
&2E(v_t,\nabla v)(t_0)+CW(t)^{(p+1)/2}+C\int_{t_0}^t s^{-1-\gamma}W(s)^{(p+1)/2}\ ds\\
&\geq \int_{\mathbf{R}^n} P(1-\epsilon C_1)(v_t^2+|\nabla v|^2)\ dx+\int_{\mathbf{R}^n} awv^2\ dx+\int_{\mathbf{R}^n}(w_t-\epsilon^{-1}t^{-(\alpha+\beta)}w)v^2\ dx\\
&\geq (1-\epsilon C_1)\int_{\mathbf{R}^n} P(v_t^2+|\nabla v|^2)\ dx+\int_{\mathbf{R}^n} awv^2 dx\\
&-(C_1+\epsilon^{-1})t^{-(\alpha+\beta)}\int_{\mathbf{R}^N} wv^2\ dx,\quad t\in[t_0,T_m).
\end{aligned}$$

This, together with Lemma (4.1.5), gives the following estimate:

$$\begin{aligned}
&(1-\epsilon C_1)\int_{\mathbf{R}^n} P(v_t^2+|\nabla v|^2)dx+\int_{\mathbf{R}^n} awv^2 dx\\
&\leq 2E(v_t,\nabla v)(t_0)+CW(t)^{(p+1)/2}+C\int_{t_0}^t s^{-1-\gamma}W(s)^{(p+1)/2}ds+C(C_1+\epsilon^{-1})t^{-(\alpha+\beta)}\\
&\left(\int_{\mathbf{R}^n} wv^2dx|_{t=t_0}+t^{\alpha+\beta}\left(E(v_t,\nabla v)(t_0)+M(t)^{(p+1)/2}+\int_{t_0}^t s^{-1-\gamma}W(s)^{(p+1)/2}ds\right)\right)\\
&\leq 2E(v_t,\nabla v)(t_0)+Ct^{-(\alpha+\beta)}\int_{\mathbf{R}^n} wv^2dx|_{t=t_0}+CE(v_t,\nabla v)(t_0)+\ CM(t)^{(p+1)/2}\\
&+C\int_{t_0}^t s^{-1-\gamma}W(s)^{(p+1)/2}ds,\quad t\in[t_0,T_m),
\end{aligned}$$

where $C>0$ is a constant independent of $\varepsilon$. By taking $\varepsilon>0$ sufficiently small,

$$\begin{aligned}
\int_{\mathbf{R}^n} P(v_t^2+|\nabla v|^2)dx+\int_{\mathbf{R}^n} awv^2dx \ &\leq\ C\left(E(v_t,\nabla v)(t_0)+\int_{\mathbf{R}^n} wv^2dx|_{t=t_0}\right)\\
&+\ CM(t)^{(p+1)/2}+C\int_{t_0}^t s^{-1-\gamma}W(s)^{(p+1)/2}ds,
\end{aligned}$$

for all $t\in[t_0,T_m)$. From this inequality and Lemma (4.1.7), we obtain the final estimate of $W(t)$:

$$W(t)\leq C_{t_0}E_u(t_0)+CM(t)^{(p+1)/2}+\frac{t_0^{-\gamma}}{\gamma}\left(\max_{0\leq s\leq t}W(s)\right)^{(p+1)/2},\quad t\in[t_0,T_m).$$

Applying Lemma (4.1.7), since $t_0 < T^*$, we have

$$M(t) \le C_{t_0}\varepsilon^2 + CM(t)^{(p+1)/2}, \quad t \in [t_0, T_m).$$

Using this estimate and standard arguments as in [47], upon possible additional decreasing of $\varepsilon > 0$, we obtain

$$M(t) \le C_{t_0}\varepsilon^2, \quad t \in [t_0, T_m). \tag{4.2.2}$$

Hence $T_m = \infty$, in other words we have a global solutions. $\square$

**Proposition 4.2.2.** *Assume that $a(t,x) = \lambda(x)\eta(t)$, where $\lambda(x)$, $\eta(t)$ are defined by (3.0.3), (3.0.4). If the weights $P$ and $w$ satisfy conditions (i)–(v) and*

$$(vi) \qquad Pw^{-3}(w_t^2 + |\nabla w|^2) \le Ca(t,x),$$

*with some constant $C$, then the solution $u$ of (4.0.1)–(4.0.2) satisfies*

$$\begin{aligned} \int a(t,x)w^{-1}u^2\, dx &\le C\varepsilon^2, \\ \int Pw^{-2}(u_t^2 + |\nabla u|^2)\, dx &\le C\varepsilon^2, \end{aligned}$$

*for $t \ge t_0$. Here $C$ depends only on the equation and weights.*

*Proof.* To get the first estimate, we use the definition of $v = uw^{-1}$. To get the second estimate, we calculate

$$v_t^2 = (-w^{-2}w_t u + w^{-1}u_t)^2 \ge \frac{1}{2}w^{-2}u_t^2 - 3w^{-4}w_t^2u^2$$

and

$$|\nabla v|^2 = |-w^{-2}u\nabla w + w^{-1}\nabla u|^2 \ge \frac{1}{2}w^{-2}|\nabla u|^2 - 3w^{-4}|\nabla w|^2u^2.$$

Thus we can write

$$\frac{1}{2}Pw^{-2}(u_t^2+|\nabla u|^2)\le P(v_t^2+|\nabla v|^2)+3Pw^{-4}(w_t^2+|\nabla w|^2)u^2.$$

Integrating this inequality over $\mathbf{R}^n$ and using (4.2.2), we obtain

$$\begin{aligned}\frac{1}{2}\int Pw^{-2}(u_t^2+|\nabla u|^2)dx &\le \int P(v_t^2+|\nabla v|^2)dx+3\int Pw^{-4}(w_t^2+|\nabla w|^2)u^2dx\\ &\le \int P(v_t^2+|\nabla v|^2)dx+C\int w^{-1}a(t,x)u^2dx\\ &\le C\varepsilon^2.\end{aligned}$$

□

*Proof.* (**Proof of Theorem (1.3.18)** Using the first estimate of Proposition (4.2.2),

$$\begin{aligned}\int a(t,x)w^{-1}u^2\ dx &\le C\varepsilon^2,\\ \int \lambda(x)\eta(t)w^{-1}u^2\ dx &\le C\varepsilon^2,\\ \int \lambda(x)w^{-1}u^2\ dx &\le C\varepsilon^2T',\\ \int \lambda(x)T^me^{\frac{S(x)}{T}}u^2\ dx &\le C\varepsilon^2T',\\ \int \lambda(x)e^{\frac{S(x)}{T}}u^2\ dx &\le C\varepsilon^2T'T^{-m},\\ \int \lambda(x)e^{(m(\lambda)-\delta)\frac{\phi(x)}{T}}u^2\ dx &\le C\varepsilon^2T^{2\delta-m(\lambda)}T',\\ \int \lambda(x)e^{(m(\lambda)-\delta)\frac{\phi(x)}{t^{\beta+1}}}u^2\ dx &\le C\varepsilon^2t^{(\beta+1)(2\delta-m(\lambda))}t^{\beta}.\end{aligned}$$

Recall that $\phi(x)$ is given by

$$\phi_0(1+|x|)^{2-\alpha}\le\phi(x)\le\phi_1(1+|x|)^{2-\alpha},\quad\text{for}\quad x\in R^n.$$

Thus $\lambda(x) \geq C(\phi(x))^{-\frac{\alpha}{2-\alpha}}$ with $C > 0$. So we have the following lower bound

$$\begin{aligned} \lambda(x) &\geq CT^{-\frac{\alpha}{2-\alpha}} \left(\frac{\phi(x)}{T}\right)^{-\frac{\alpha}{2-\alpha}} \\ &\geq CT^{-\frac{\alpha}{2-\alpha}} e^{-\delta\frac{\phi(x)}{T}} \\ &\geq Ct^{-(\beta+1)\frac{\alpha}{2-\alpha}} e^{-\delta\frac{\phi(x)}{t^{(\beta+1)}}}, \end{aligned} \tag{4.2.3}$$

where $C > 0$ and $t \geq t_0$ is sufficiently large. We can complete the decay estimate for the $L^2$ norm of $u$ by substituting this lower bound for $\lambda(x)$:

$$\begin{aligned} \int e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} u^2 &\leq C\varepsilon^2 t^{\beta+(\beta+1)(2\delta-m(\lambda))+(\beta+1)(\frac{\alpha}{2-\alpha})}, \\ \int e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} u^2 &\leq C\varepsilon^2 t^{\beta+(\beta+1)(2\delta-m(\lambda)+\frac{\alpha}{2-\alpha})}. \end{aligned}$$

To show the decay estimate for the energy of $u$, we begin with

$$\begin{aligned} \int Pw^{-2}(u_t^2 + |\nabla u|^2)\, dx &\leq C\varepsilon^2, \\ \int \frac{3}{4}\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1} \frac{T^m}{T'} e^{\frac{S(x)}{T}} (u_t^2 + |\nabla u|^2) dx &\leq C\varepsilon^2, \\ \int \left(\frac{1}{T} + \frac{\phi(x)}{T^2}\right)^{-1} e^{(m(\lambda)-\delta)\frac{\phi(x)}{T}} (u_t^2 + |\nabla u|^2) dx &\leq C\varepsilon^2 T' T^{2\delta - m(\lambda)}. \end{aligned}$$

Now we observe that

$$\left(\frac{1}{T} + \frac{\phi(x)}{T^2}\right)^{-1} = T\left(1 + \frac{\phi(x)}{T}\right)^{-1} \geq Te^{-\delta\frac{\phi(x)}{T}},$$

so we have

$$\begin{aligned} \int e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} (u_t^2 + |\nabla u|^2) dx &\leq C\varepsilon^2 t^\beta t^{(\beta+1)(2\delta-m(\lambda))} t^{-(\beta+1)}, \\ \int e^{(m(\lambda)-2\delta)\frac{\phi(x)}{t^{\beta+1}}} (u_t^2 + |\nabla u|^2) dx &\leq C\varepsilon^2 t^{(\beta+1)(2\delta-m(\lambda))-1}. \end{aligned}$$

Finally, we can show the decay estimate for the $L^{p+1}$ norm of $u$. We use Lemma (4.1.2) together with inequality (4.2.2), and the definition of weights $P$ and $w$:

$$\begin{aligned}
\int_{\mathbf{R}^n} P w^{p-1} |v|^{p+1}\, dx &\leq C\varepsilon^2 t^{-\xi}, \\
\int_{\mathbf{R}^n} \frac{3}{4}\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1} \frac{w}{T'} w^{p-1} \left|\frac{u}{w}\right|^{p+1} dx &\leq C\varepsilon^2 t^{-\xi}, \\
\int_{\mathbf{R}^n} \frac{3}{4} T\left(6 + \frac{S(x)}{T}\right)^{-1} \frac{w^{-1}}{T'} |u|^{p+1}\, dx &\leq C\varepsilon^2 t^{-\xi}, \\
\int_{\mathbf{R}^n} \frac{T}{T'} w^{-1} |u|^{p+1}\, dx &\leq C\varepsilon^2 t^{-\xi}.
\end{aligned}$$

Since $T \sim t^{\beta+1}$ and $T' \sim t^{\beta}$, we have

$$\begin{aligned}
\int_{\mathbf{R}^n} w^{-1} |u|^{p+1}\, dx &\leq C\varepsilon^2 t^{-\xi-1}, \\
\int_{\mathbf{R}^n} e^{\frac{S(x)}{T}} T^{mp} |u|^{p+1}\, dx &\leq C\varepsilon^2 t^{-\xi-1}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-\delta)\frac{\phi(x)}{T}} |u|^{p+1}\, dx &\leq C\varepsilon^2 T^{(\delta-m(\lambda))p} t^{-\xi-1}, \\
\int_{\mathbf{R}^n} e^{(m(\lambda)-\delta)\frac{\phi(x)}{t^{\beta+1}}} |u|^{p+1}\, dx &\leq C\varepsilon^2 t^{-(\xi+1)+p(\beta+1)(\delta-m(\lambda))}.
\end{aligned}$$

$\square$

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# Bibliography


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# Appendices

# Appendix A

**Lemma A.3.** *Let $P$ and $w$ be defined by (2.3.2) and $\hat{c}$ be defined at (2.1.11). Then*

$$\begin{aligned} \hat{c} &\geq \left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)T'' + \left(\frac{m}{T^2}-\frac{2S(x)}{T^3}\right){T'}^2 - \frac{|\nabla S(x)|^2}{T^2} + \frac{\Delta S(x)}{T} \\ &+ \left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)\lambda(x). \end{aligned} \tag{A.4}$$

$$\hat{c} \geq \frac{-mT''+\delta\lambda(x)}{T} + \frac{S(x)T''+k\delta\lambda(x)S(x)-\frac{2S(x){T'}^2}{T}}{T^2}. \tag{A.5}$$

*Proof.* First calculate the first and second order derivatives of $w$ :

$$\begin{aligned} w_t &= \left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)T'w, \\ w_{tt} &= \left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)T''w + \left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)^2{T'}^2w + \left(\frac{m}{T^2}-\frac{2S(x)}{T^3}\right){T'}^2w, \\ \nabla w &= -\frac{\nabla S(x)}{T}\ w, \quad \Delta w = \left(-\frac{\Delta S(x)}{T}+\frac{|\nabla S(x)|^2}{T^2}\right)w. \end{aligned} \tag{A.6}$$

Substitute $w_{tt}$, $\Delta w$, and $w_t$ into (2.1.11),

$$\begin{aligned}
\hat{c} &= \left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)T''+\left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)^2 T'^2+\left(\frac{m}{T^2}-\frac{2S(x)}{T^3}\right)T'^2 \\
&- \frac{|\nabla S(x)|^2}{T^2}+\frac{\Delta S(x)}{T}+\left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)\lambda(x), \\
&\geq \left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)T''+\left(\frac{m}{T^2}-\frac{2S(x)}{T^3}\right)T'^2-\frac{|\nabla S(x)|^2}{T^2}+\frac{\Delta S(x)}{T} \\
&+ \left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)\lambda(x).
\end{aligned}$$

Using Lemma (2.3.1)

$$\begin{aligned}
\hat{c} &= \frac{-mT''+(m+\delta)\lambda(x)-m\lambda(x)}{T} \\
&+ \frac{S(x)T''+mT'^2-\lambda(x)S(x)+k\delta\lambda(x)S(x)+\lambda(x)S(x)}{T^2}-\frac{2S(x)T'^2}{T^3} \\
&= \frac{-mT''+\delta\lambda(x)}{T}+\frac{S(x)T''+k\delta\lambda(x)S(x)-\frac{2S(x)T'^2}{T}}{T^2}.
\end{aligned}$$

□

**Lemma A.4.** *Let $P$ and $w$ be defined by (2.3.2). Then*

$$\begin{aligned}
&(1)\quad \frac{P_t}{P} \leq T'\left(\frac{-m+1}{T}+\frac{4}{3}\frac{S(x)}{T^2}-\frac{T''}{T'^2}\right), \\
&(2)\quad -P_t+w \geq \left(\frac{m+7}{T}+\frac{T''}{T'^2}\right)T'P, \\
&(3)\quad -P_t+2aP+4P(\ln w)_t-2w \geq \left(\frac{-3m-17}{T}+\frac{T''}{T'^2}+\frac{2\lambda(x)\eta(t)}{T'}\right)T'P, \\
&(4)\quad (\nabla P-2P\nabla\ln w)^2 = \frac{(5T+S(x))^2|\nabla S(x)|^2}{T^2(6T+S(x))^2}P^2, \\
&(5)\quad Pw^{-3}\left(w_t^2+|\nabla w|^2\right) \leq C\left(\frac{m^2T'}{T}+\frac{S(x)}{T^2}T'+\lambda(x)\eta(t)\right).
\end{aligned}$$

*Proof.* (1) We calculate $P_t/P = (\ln P)_t$ using the definition of the weight $P$ as in (2.3.2):

$$\begin{aligned}
\frac{P_t}{P} &= \left(\ln\frac{3}{4} - \ln\left(\frac{6}{T} + \frac{S(x)}{T^2}\right) + \ln\frac{1}{T'} + \ln w\right)_t \\
&= -\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1}\left(\frac{-6T'}{T^2} + \frac{-2TT'S(x)}{T^4}\right) + \frac{-T'T''}{T'^2} + \frac{w_t}{w} \\
&= -T'\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1}\left(\frac{-6}{T^2} - \frac{2S(x)}{T^3}\right) - \frac{T''}{T'} + \frac{1}{w}T'w\left(\frac{S(x)}{T^2} - \frac{m}{T}\right) \\
&= T'\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1}\left(\frac{6}{T^2} + \frac{2S(x)}{T^3}\right) - \frac{T''}{T'} + T'\left(\frac{S(x)}{T^2} - \frac{m}{T}\right) \\
&= TT'\left(6 + \frac{S(x)}{T}\right)^{-1}\left(\frac{6}{T^2} + \frac{2S(x)}{T^3}\right) - \frac{T''}{T'} + T'\left(\frac{S(x)}{T^2} - \frac{m}{T}\right) \\
&= T'\left(6 + \frac{S(x)}{T}\right)^{-1}\left(\frac{6}{T} + \frac{2S(x)}{T^2}\right) - \frac{T''}{T'} + T'\left(\frac{S(x)}{T^2} - \frac{m}{T}\right) \\
&\leq \frac{T'}{6}\left(\frac{6}{T} + \frac{2S(x)}{T^2}\right) - \frac{T''}{T'} + T'\left(\frac{S(x)}{T^2} - \frac{m}{T}\right) \\
&= T'\left(\frac{1}{T} + \frac{S(x)}{3T^2}\right) - \frac{T''}{T'} + T'\left(\frac{S(x)}{T^2} - \frac{m}{T}\right) \\
&= \frac{T'(1-m)}{T} + \frac{T'(S(x) + 3S(x))}{3T^2} - \frac{T''}{T'} \\
&= T'\left(\frac{1-m}{T} + \frac{4S(x)}{3T^2} - \frac{T''}{T'^2}\right).
\end{aligned}$$

(2) Using the weights $P$ and $w$ as defined by (2.3.2)

$$\begin{aligned}
-P_t + w &= \left(-\frac{P_t}{P} + \frac{w}{P}\right)P \\
&\geq \left(\frac{m-1}{T} - \frac{4S(x)}{3T^2} + \frac{T''}{T'^2} + \frac{8}{T} + \frac{4S(x)}{3T^2}\right)T'P \\
&= \left(\frac{m+7}{T} + \frac{T''}{T'^2}\right)T'P.
\end{aligned}$$

(3) $-P_t + 2aP + 4P(\ln w)_t - 2w = \left(-\frac{P_t}{P} + 2\lambda(x)\eta(t) + 4\frac{w_t}{w} - 2\frac{w}{P}\right)P.$

Now to estimate $-P_t/P$ apply (1) above:

$$-\frac{P_t}{P} \geq T'\left(\frac{m-1}{T} - \frac{4}{3}\frac{S(x)}{T^2} + \frac{T''}{T'^2}\right).$$

Also

$$
\begin{aligned}
(\ln w)_t &= \left(-m\ln T - \frac{S(x)}{T}\right)_t \\
&= -\frac{mT'}{T} + \frac{S(x)T'}{T^2} \\
&= \left(-\frac{m}{T} + \frac{S(x)}{T^2}\right)T'. \\
-\frac{2w}{P} &= -2\left(\frac{4}{3}\right)\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)T'. \\
-P_t + 2aP + 4P(\ln w)_t - 2w &\geq \left(\frac{m-1}{T} - \frac{4S(x)}{3T^2} + \frac{T''}{T'^2}\right. \\
&+ \left.\frac{2\lambda(x)\eta(t)}{T'} - \frac{4m}{T} + \frac{4S(x)}{T^2} - \frac{8S(x)}{3T^2} - \frac{16}{T}\right)T'P \\
&= \left(\frac{-3m-17}{T} + \frac{T''}{T'^2} + \frac{2\lambda(x)\eta(t)}{T'}\right)T'P.
\end{aligned}
$$

(4)

$$
\begin{aligned}
\nabla P - 2P\nabla \ln w &= \left(\frac{\nabla P}{P} - 2\nabla \ln w\right) P \\
&= \left(\nabla \ln P - 2\nabla \ln w\right) P. \\
\ln P &= \ln\frac{3}{4} - \ln\left(\frac{6}{T} + \frac{S(x)}{T^2}\right) - \ln T' + \ln w. \\
\nabla \ln P &= -\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1} \frac{\nabla S(x)}{T^2} + \nabla \ln w. \\
\nabla P - 2P\nabla \ln w &= \left(-\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1} \frac{\nabla S(x)}{T^2} + \nabla \ln w - 2\nabla \ln w\right) P \\
&= \left(-\left(\frac{6}{T} + \frac{S(x)}{T^2}\right)^{-1} \frac{\nabla S(x)}{T^2} - \nabla \ln w\right) P \\
&= \left(-\frac{T^2}{6T + S(x)} \frac{\nabla S(x)}{T^2} + \frac{\nabla S(x)}{T}\right) P \\
&= \left(-\frac{\nabla S(x)}{6T + S(x)} + \frac{\nabla S(x)}{T}\right) P \\
&= \frac{(5T + S(x))\nabla S(x)}{T(6T + S(x))} P. \\
(\nabla P - 2P\nabla \ln w)^2 &= \frac{(5T + S(x))^2 |\nabla S(x)|^2}{T^2(6T + S(x))^2} P^2.
\end{aligned}
$$

(5)

$$
\begin{aligned}
Pw^{-3}\left(w_t^2+|\nabla w|^2\right) &= \frac{P}{w}\left(\left(\frac{w_t}{w}\right)^2+\frac{|\nabla w|^2}{w^2}\right).\\
\frac{P}{w} &= \frac{3}{4}\left(\frac{6}{T}+\frac{S(x)}{T^2}\right)^{-1}\frac{1}{T'}\\
&= \frac{3}{4}\left(\frac{6T+S(x)}{T^2}\right)^{-1}\frac{1}{T'}\\
&= \frac{3}{4}\left(\frac{T^2}{6T+S(x)}\right)\frac{1}{T'}.\\
\left(\frac{w_t}{w}\right)^2 &= \left(-\frac{m}{T}+\frac{S(x)}{T^2}\right)^2 T'^2 \le \left(\frac{2m^2}{T^2}+\frac{2S(x)^2}{T^4}\right)T'^2.\\
\frac{|\nabla w|^2}{w^2} &= \frac{|\nabla S(x)|^2}{T^2} \le \frac{\lambda(x)S(x)}{T^2}.\\
\frac{P}{w}\left(\left(\frac{w_t}{w}\right)^2+\frac{|\nabla w|^2}{w^2}\right) &\le \frac{3}{4}\left(\frac{T^2}{6T+S(x)}\right)\frac{1}{T'}\left(\left(\frac{2m^2}{T^2}+\frac{2S(x)^2}{T^4}\right)T'^2+\frac{\lambda(x)S(x)}{T^2}\right)\\
&\le \frac{3}{4}\left(\frac{T^2}{6T}\frac{1}{T'}\frac{2m^2}{T^2}T'^2+\frac{T^2}{S(x)}\frac{1}{T'}\frac{2S(x)^2}{T^4}T'^2+\frac{T^2}{S(x)}\frac{1}{T'}\lambda(x)\frac{S(x)}{T^2}\right)\\
&\le C\left(\frac{m^2T'}{T}+\frac{S(x)}{T^2}T'+\frac{\lambda(x)}{T'}\right)\\
&= C\left(\frac{m^2T'}{T}+\frac{S(x)}{T^2}T'+\lambda(x)\lambda(t)\right).
\end{aligned}
$$

$\square$

# Appendix B

## Derivation of the Weighted Energy

Here we derive a modified equation for $v = w^{-1}u$, and the weighted energy identities.

Let us consider a general first–order perturbation of the semi–linear wave equation:

$$u_{tt} - \Delta u + au_t + b \cdot \nabla u + cu + k|u|^{p-1}u = 0, \tag{B.7}$$

where the coefficients $a$, $b = (b_1, b_2, \ldots, b_n)$, $c \geq 0$ are in $C^1(\mathbf{R}_+, \mathbf{R}^n)$ functions, and $k \in \{-1, 0, 1\}$. Using the following transformation $v = w^{-1}u$ where $w$ is the approximate solution of (3.1.2), we have the following transformed equation:

$$v_{tt} - \Delta v + \hat{a}v_t + \hat{b} \cdot \nabla v + \hat{c}v + k\hat{h}|v|^{p-1}v = 0, \tag{B.8}$$

where the coefficients are given by

$$\begin{aligned}
\hat{a} &= a(t,x) + 2w^{-1}w_t, \\
\hat{b} &= b - 2w^{-1} \cdot \nabla w, \\
\hat{c} &= w^{-1}(w_{tt} - \Delta w + a(t,x)w_t + b \cdot \nabla w + cw), \\
\hat{h} &= w^{p-1}.
\end{aligned} \tag{B.9}$$

To show this, using the following simple calculation we have

$$\begin{aligned}
u_t &= wv_t + vw_t, \\
u_{tt} &= wv_{tt} + 2v_t w_t + vw_{tt}, \\
\nabla u &= w \cdot \nabla v + v \cdot \nabla w, \\
\triangle u &= w\triangle v + 2\nabla v \cdot \nabla w + v\triangle w.
\end{aligned}$$

Substitute these equations in (B.7) so we have:

$$\begin{aligned}
0 &= wv_{tt} + 2v_t w_t + vw_{tt} - w\triangle v - 2\nabla v \cdot \nabla w - v\triangle w + a(t,x)wv_t \\
&+ a(t,x)vw_t + wb.\nabla v + bv \cdot \nabla w + cwv + k|w|^{p-1}w|v|^{p-1}v, \\
0 &= w\Bigg(v_{tt} + v_t\Big(2w_t w^{-1} + a(t,x)\Big) + vw^{-1}(w_{tt} - \triangle w + a(t,x)w_t + b \cdot \nabla w + cw) \\
&- \triangle v + \nabla v\Big(b - 2w^{-1} \cdot \nabla w\Big) + kw^{p-1}|v|^{p-1}v\Bigg).
\end{aligned}$$

The transformed equation will be (B.8) with coefficients as given by (B.9). To find the weighted energy multiply the transformed equation (B.9) by $Pv_t + wv$ where $P, w \in C^2((0,\infty) \times \mathbf{R^n})$, are the weights that have to be defined.

**Proposition B.5.** *Let $u \in C((0,\infty), H^2(\mathbf{R}^n)) \cap C^1((0,\infty), H^1(\mathbf{R}^n))$ be a solution of (B.7) with compact supported data*

$$u_0 \in H^2(\mathbf{R}^n), \qquad u_1 \in H^1(\mathbf{R}^n).$$

*and $\hat{h} \geq 0$. For any pair of $C^2$-functions $P$ and $w$, we have the equality*

$$\frac{d}{dt}A(v_t, \nabla v, v) + F(v_t, \nabla v) + G(v) + H(v) = 0, \tag{B.10}$$

*where*

$$
\begin{aligned}
A(v_t, \nabla v, v) &= \frac{1}{2}\int \left( P(v_t^2 + |\nabla v|^2) + 2wv_tv + (\hat{c}P - w_t + \hat{a}w)v^2 \right) \, dx \\
&\quad + \frac{k}{p+1}\int P\hat{h}|v|^{p+1} \, dx, \\
F(v_t, \nabla v) &= \frac{1}{2}\int (-P_t + 2\hat{a}P - 2w)v_t^2 \, dx \\
&\quad + \int (\nabla P + \hat{b}P) \cdot v_t \nabla v \, dx \\
&\quad + \frac{1}{2}\int (-P_t + 2w)|\nabla v|^2 \, dx, \\
G(v) &= \frac{1}{2}\int [w_{tt} - \Delta w - (\hat{a}w)_t - \nabla \cdot (w\hat{b}) + 2\hat{c}w - (\hat{c}P)_t]v^2 \, dx, \\
H(v) &= k\int \left( w\hat{h} - \frac{1}{p+1}(P\hat{h})_t \right) |v|^{p+1} \, dx.
\end{aligned}
$$

*the coefficients $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{h}$ are given in (B.9)*

*Proof.* Given $P$ and $w$ in $C^2((0, \infty) \times \mathbf{R}^n)$, multiply equation (3.1.8) by $Pv_t + wv$.

$$
(v_{tt} - \Delta v + \hat{a}v_t + \hat{b} \cdot \nabla v + \hat{c}v + k\hat{h}|v|^{p-1}v)(Pv_t + wv) = 0.
$$

**Step I.** We multiply equation (B.8) by $v_t$ and rearrange the terms as follows:

$$
\frac{d}{dt}\left( \frac{v_t^2 + |\nabla v|^2}{2} \right) - \nabla \cdot (v_t \nabla v) + \hat{a}v_t^2 + v_t\hat{b} \cdot \nabla v + \hat{c}\frac{d}{dt}\left( \frac{v^2}{2} \right) + k\hat{h}\frac{d}{dt}\left( \frac{|v|^{p+1}}{p+1} \right) = 0.
$$

**Step II.** Let us multiply the identity from Step I by $P$ and integrate over $\mathbf{R}^n$.

$$
\begin{aligned}
0 &= \frac{d}{dt}\int P\frac{v_t^2 + |\nabla v|^2}{2} \, dx - \int P_t\frac{v_t^2 + |\nabla v|^2}{2} \, dx + \int \nabla P \cdot v_t \nabla v \, dx \\
&\quad + \int (P\hat{a}v_t^2 + v_tP\hat{b} \cdot \nabla v) \, dx + \frac{d}{dt}\int P\hat{c}\frac{v^2}{2} \, dx - \int (P\hat{c})_t\frac{v^2}{2} \, dx \\
&\quad + k\frac{d}{dt}\int P\hat{h}\frac{|v|^{p+1}}{p+1} \, dx - k\int (P\hat{h})_t\frac{|v|^{p+1}}{p+1} \, dx.
\end{aligned}
$$

**Step III.** We now multiply equation (B.8) by $v$ and again rearrange the terms:

$$\frac{d}{dt}(v_t v) - \nabla\cdot(v\nabla v) - v_t^2 + |\nabla v|^2 + \hat{a}\frac{d}{dt}\left(\frac{v^2}{2}\right) + \hat{b}\cdot\nabla\frac{v^2}{2} + \hat{c}v^2 + k\hat{h}|v|^{p+1} = 0.$$

**Step IV.** We also multiply the identity from Step III by $w$ and integrate over $\mathbf{R}^n$.

$$\begin{aligned} 0 \;=\;& \frac{d}{dt}\int wv_t v\ dx - \int w_t v_t v\ dx + \int \nabla w\cdot v\nabla v\ dx \\ &+ \int (w|\nabla v|^2 - wv_t^2)\ dx + \frac{d}{dt}\int w\hat{a}\frac{v^2}{2}\ dx - \int (w\hat{a})_t\frac{v^2}{2}\ dx \\ &- \int \nabla\cdot(w\hat{b})\frac{v^2}{2}\ dx + \int w\hat{c}v^2\ dx + k\int w\hat{h}|v|^{p+1}\ dx. \end{aligned}$$

Using

$$w_t v_t v = \frac{1}{2}(w_t v^2)_t - \frac{1}{2}w_{tt}v^2, \quad \nabla w\cdot v\nabla v = \frac{1}{2}\nabla\cdot(v^2\nabla w) - \frac{1}{2}v^2\Delta w.$$

We can rewrite the above identity as

$$\begin{aligned} 0 \;=\;& \frac{d}{dt}\int (wv_t v - \frac{1}{2}w_t v^2)\ dx + \int \frac{1}{2}w_{tt}v^2\ dx - \int \frac{1}{2}\Delta w v^2\ dx \\ &+ \int (w|\nabla v|^2 - wv_t^2)\ dx + \frac{d}{dt}\int w\hat{a}\frac{v^2}{2}\ dx - \int (w\hat{a})_t\frac{v^2}{2}\ dx \\ &- \int \nabla\cdot(w\hat{b})\frac{v^2}{2}\ dx + \int w\hat{c}v^2\ dx + k\int w\hat{h}|v|^{p+1}\ dx. \end{aligned}$$

**Step V.** We add the final identities from Step II, Step IV and combine similar terms. This establishes the following result:

$$\frac{d}{dt}A(v_t, \nabla v, v) + F(v_t, \nabla v) + G(v) + H(v) = 0.$$

$\square$

# Appendix C

## The Poisson Equation in $\mathbf{R}^n$

To establish the decay estimates in Corollary (1.3.3) and (1.3.15), Studying the Poisson equation

$$\Delta\phi(x) = \lambda(x), \quad x \in \mathbf{R}^n, \tag{C.11}$$

with a positive radial $\lambda(x)$ in $n \geq 3$. The problem is to find a radial solution $\phi(x)$ in the class of functions satisfying conditions $(a1) - (a3)$. Another difficulty is to find the asymptotic behavior of $\phi(x)$ as $|x| \to \infty$ and compute the decay rate $m(\lambda)$.

We rely on the radial symmetry of $\lambda(x)$ to simplify the solution, since the Poisson equation for radial functions becomes an ODE:

$$\frac{d^2\phi}{dr^2} + \frac{n-1}{r}\frac{d\phi}{dr} = \lambda(r), \quad r = |x|,$$

where we write $\lambda(r)$ instead of $\lambda(r\omega)$, $\omega \in \mathbf{S}^{n-1}$. Multiply with $r^{n-1}$ and rewrite the left side to obtain

$$\frac{d}{dr}\left(r^{n-1}\frac{d\phi}{dr}\right) = r^{n-1}\lambda(r).$$

Integrate this equation on $[0, r]$, $r > 0$. We have that

$$\frac{d\phi(r)}{dr} = r^{1-n} \int_0^r \tau^{n-1} \lambda(\tau) \; d\tau. \tag{C.12}$$

Integrate the resulting equation (C.12) with $\phi(0) = 0$, the result is,

$$\begin{aligned}
\int_0^r \frac{d\phi(s)}{ds} \; ds &= \int_0^r s^{1-n} \int_0^s \tau^{n-1} \lambda(\tau) \; d\tau \; ds \\
\phi(r) - \phi(0) &= \int_0^r \int_\tau^r s^{1-n} \tau^{n-1} \lambda(\tau) \; ds \; d\tau \\
\phi(r) &= \int_0^r \int_\tau^r \frac{d}{ds} \left( \frac{s^{2-n}}{2-n} \right) \tau^{n-1} \lambda(\tau) \; ds \; d\tau \\
&= \frac{1}{2-n} \int_0^r \left( r^{2-n} - \tau^{2-n} \right) \tau^{n-1} \lambda(\tau) \; d\tau \\
&= -\frac{1}{n-2} \int_0^r \left( \frac{r^{2-n}}{\tau^{2-n}} - 1 \right) \tau^{2-n} \tau^{n-1} \lambda(\tau) \; d\tau \\
&= -\frac{1}{n-2} \int_0^r \left( \frac{r^{2-n}}{\tau^{2-n}} - 1 \right) \tau \lambda(\tau) \; d\tau \\
&= \frac{1}{n-2} \int_0^r \left( 1 - \frac{\tau^{n-2}}{r^{n-2}} \right) \tau \lambda(\tau) \; d\tau. \qquad \text{(C.13)}
\end{aligned}$$

This solution satisfies $\phi(0) = 0$ and $\frac{d\phi(0)}{dr} = 0$. Recall that $\phi(r)$ means $\phi(r\omega)$ with $\omega \in \mathbf{S}^{n-1}$. We will use the above formulas for $\phi(r)$ to show the following.

**Proposition C.6.** *Assume that $\lambda(x)$ depends on $r = |x|$ and satisfies*

$$\lambda_0 (1 + |x|)^\alpha \le \lambda(x) \le \lambda_1 (1 + |x|)^\alpha, \quad \alpha \in (0, \infty),$$

*where $\lambda_0$ and $\lambda_1$ are positive constants. Then the solution $\phi(x)$ of equation (C.11), defined in (C.13), satisfies*

$$\phi_0 (1 + |x|)^{2+\alpha} \le \phi(x) \le \phi_1 (1 + |x|)^{2+\alpha}, \quad x \in \mathbf{R}^n,$$

$$\phi_2 |x| \le |\nabla \phi(x)| \le \phi_3 (1 + |x|)^{1+\alpha}, \qquad |x| \ge 1,$$

*for some* $\phi_0, \dots, \phi_3 > 0$.

*Proof.* To estimate $\phi(x)$ we substitute the lower and upper bounds of $\lambda(x)$ into formula (C.13).

To derive the upper bound we use $\lambda(x) \leq \lambda_1(1+|x|)^\alpha$,

$$
\begin{aligned}
\phi(r) &\leq \frac{1}{n-2}\int_0^r \left(1-\frac{\tau^{n-2}}{r^{n-2}}\right)\tau\lambda_1(1+\tau)^\alpha d\tau \\
&\leq \frac{\lambda_1}{n-2}\int_0^r \tau(1+\tau)^\alpha d\tau \\
&\leq \frac{\lambda_1}{n-2}\int_0^r (1+\tau)^{\alpha+1} d\tau \\
&\leq \phi_1(1+r)^{\alpha+2}, \text{ where } \ \phi_1 > 0.
\end{aligned}
$$

To derive the lower bound we use $\lambda(x) \geq \lambda_0(1+|x|)^\alpha$,

$$
\begin{aligned}
\phi(r) &\geq \frac{1}{n-2}\int_0^r \tau\left(1-\frac{\tau^{n-2}}{r^{n-2}}\right)\lambda_0(1+\tau)^\alpha d\tau \\
&\geq \frac{\lambda_0}{n-2}\int_0^{r/2} \tau\left(1-\frac{\tau^{n-2}}{r^{n-2}}\right)(1+\tau)^\alpha d\tau \\
&\geq \frac{\lambda_0}{n-2}\int_0^{r/2} \frac{1}{2}\tau(1+\tau)^\alpha d\tau \\
&= \frac{\lambda_0}{2(n-2)}\int_1^{1+r/2} (u-1)u^\alpha du \\
&= \frac{\lambda_0}{2(n-2)}\int_1^{1+r/2} (u^{\alpha+1}-u^\alpha) du \\
&\geq \frac{\lambda_0}{2(n-2)}\int_1^{1+r/2} u^{\alpha+1}\ du \\
&\geq \frac{\lambda_0}{2^{\alpha+3}(n-2)(\alpha+2)}(2+r)^{\alpha+2} \\
&\geq \phi_0(1+r)^{\alpha+2}, \text{ where } \ \phi_0 > 0.
\end{aligned}
$$

Hence

$$\phi_0(1+|x|)^{2+\alpha} \leq \phi(x) \leq \phi_1(1+|x|)^{2+\alpha}.$$

To show the second inequality, we substitute the lower and upper bounds of $\lambda(x)$ into expression (C.12).

$$\begin{aligned}
\frac{d\phi(r)}{dr} &\leq r^{1-n}\int_0^r \tau^{n-1}\lambda_1(1+\tau)^\alpha d\tau, \quad r \geq 0 \\
&\leq r^{1-n}\int_0^r r^{n-1}\lambda_1(1+\tau)^\alpha d\tau \\
&= \frac{\lambda_1}{\alpha+1}(1+\tau)^{\alpha+1}|_0^r \\
&\leq \phi_3(1+r)^{\alpha+1}, \text{ where } \phi_3 > 0.
\end{aligned}$$

to estimate the lower bound we use $\lambda_0 \leq \lambda_0(1+|x|)^\alpha \leq \lambda(x)$

$$\begin{aligned}
\frac{d\phi(r)}{dr} &\geq r^{1-n}\int_0^r \tau^{n-1}\lambda_0 d\tau \\
&= \frac{\lambda_0}{n}r^{1-n}r^n \\
&= \phi_2 r, \text{ where } \phi_2 > 0.
\end{aligned}$$

□

Proof of Proposition (3.1.1)

*Proof.* It follows from (C.12) and $\lambda(r) = \lambda_2 r^\alpha + o(r^\alpha)$ that

$$\begin{aligned}
\frac{d\phi(r)}{dr} &= r^{1-n}\int_0^r \tau^{n-1}[\lambda_2\tau^\alpha + o(\tau^\alpha)]\, d\tau, \quad r \to \infty. \\
&= \lambda_2 r^{1-n}\frac{r^{n+\alpha}}{n+\alpha} + o(r^{1+\alpha}) \\
&= \frac{\lambda_2}{n+\alpha}r^{1+\alpha} + o(r^{1+\alpha}) \\
\phi(r) &= \frac{\lambda_2}{(n+\alpha)(\alpha+2)}r^{2+\alpha} + o(r^{2+\alpha}).
\end{aligned}$$

As $r \to \infty$. Now the definition of $m(\lambda)$ yields

$$\begin{aligned} m(\lambda) &= \frac{\lambda_2 r^\alpha \lambda_2 r^{2+\alpha}}{(n+\alpha)(\alpha+2)} \cdot \frac{(n+\alpha)^2}{\lambda_2^2 r^{2(1+\alpha)}} \\ &= \frac{n+\alpha}{2+\alpha}. \end{aligned}$$

□

**Corollary C.7.** *Assume that $\lambda(x)$ depends on $r = |x|$ and satisfies*

$$\lambda_0(1+|x|)^\alpha \leq \lambda(x) \leq \lambda_1(1+|x|)^\alpha, \quad \alpha \in (0, \infty),$$

*where $\lambda_0$ and $\lambda_1$ are positive constants. Then $S(x)$ satisfies*

$$\tfrac{\lambda_0(1+|x|)^{2+\alpha}}{(n+\alpha)(2+\alpha)} \leq S(x) \leq \tfrac{\lambda_1(1+|x|)^{2+\alpha}}{(n+\alpha)(2+\alpha)}, \quad x \in \mathbf{R}^n.$$

*Proof.* Is similar to the previous proof. □

# Vita

Maisa Khader was born in Ramallah, Palestine. She lived in Ramallah until she graduated from Al-Quds University with her Bachelor of Science in Mathematics in 1995. She earned a scholarship from Karim Rida Said Foundation (KRSF) to work towards her masters degree. She moved to Washington D.C., USA and earned her masters degree in Statistics in 1996 from the American University. She moved back to Ramallah and worked in the Palestinian Census Bureau for six months. She became Head of the Statistics Department for the Ministry of Higher Education in 1997. While working there she published the Statistics Year Book for the higher education institutes in the West Bank and Gaza Strip. In 1997 she moved to Mobile, Alabama, USA and earned a masters degree in Mathematics from the University of South Alabama in 1999. After earning her MS degree, her family moved to Cleveland, Ohio where she worked as a substitute teacher at the St. Joseph Academy. After one year she moved to Amman Jordan. Here she worked at Al-Manhal International School teaching mathematics for A-level GCSE/British National Curriculum. After that she worked at the Agricultural Credit Corporation in Amman in the Statistics Department where she also published the Statistics Year Book for the Corporation. Eight months later she moved to the United States to attend the Mathematics Graduate Program at The University of Tennessee. While working as a Graduate Teaching Associate in the Math Department, she earned her Doctorate in Mathematics in 2009.